Advanced Quantum Field Theory SS 2023

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Sheet 09: Infrared Singularities

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1 QCD corrections to $q\bar{q}$ production in e^-e^+ annihilation

In this problem, we will fill in the gaps of the computation of the leading QCD corrections to the process

$$e^-(p_1) + e^+(p_2) \to q(p_3) + \bar{q}(p_4) \tag{1}$$

studied in the lecture. It contains a number of bonus questions, which you may attempt to solve at your own discretion.

All fermions are assumed to be massless and we neglect the contribution of Z-boson exchange for simplicity. Working in dimensional regularization with $D = 4 - 2\varepsilon$ and the $\overline{\text{MS}}$-scheme, the leading result for the corresponding cross section was derived as

$$\sigma^{(0)} = \frac{4\pi\alpha^2}{3s} Q_q^2 N_c C(\varepsilon), \quad C(\varepsilon) = \frac{3(1-\varepsilon)^2 \Gamma(1-\varepsilon) e^{\varepsilon\gamma_E}}{(3-2\varepsilon)\Gamma(2-2\varepsilon)} \left(\frac{s}{\mu^2}\right)^{-\varepsilon} = 1 + \mathcal{O}(\varepsilon) , \tag{2}$$

where N_c denotes the number of colours, Q_q is the electric charge of the quark q and $s = (p_1+p_2)^2$.

1.1 Virtual QCD correction

The leading virtual correction in QCD comes from the exchange of a gluon between the quark and anti-quark. The corresponding amplitude gives rise to the following tensorial one-loop integral

$$I^\mu = \tilde{\mu}^{2\varepsilon} \int \frac{d^Dk}{(2\pi)^D} \frac{\gamma_\nu(\not{k}+\not{p}_3)\gamma^\mu(\not{k}-\not{p}_4)\gamma^\nu}{(k^2+i\delta^+)((k+p_3)^2+i\delta^+)((k-p_4)^2+i\delta^+)} \equiv \gamma_\nu\gamma_\rho\gamma^\mu\gamma_\lambda\gamma^\nu \tilde{I}^{\rho\lambda} , \tag{3}$$

where as in the lecture we put

$$\tilde{\mu} = \mu\sqrt{\frac{e^{\gamma_E}}{4\pi}} .$$

To calculate the virtual correction to the cross section, you used in the lecture that this integral can be expressed through a one-loop massless bubble integral which you computed in the first exercise sheet. We will now prove this result or more precisely, we will show that

$$\bar{u}(p_3) I^\mu u(p_4) = \bar{u}(p_3)\gamma^\mu u(p_4) \frac{(D^2-7D+16)}{D-4} \text{Bub}(s;D) , \tag{4}$$

where

$$\text{Bub}(s;D) = \tilde{\mu}^{2\varepsilon} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2+i\delta^+)((k-p_3-p_4)^2+i\delta^+)} \tag{5}$$

In the following, we omit Feynman's $i\delta^+$-prescription for ease of typing.

1. Explain why

$$\int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2(k+p_3)^2} = \int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2(k-p_4)^2} = 0 . \tag{6}$$

2. Prove that the bubble with a numerator can be reduced to the bubble without a numerator:[1]

$$\widetilde{\mu}^{2\varepsilon}\int\frac{d^Dk}{(2\pi)^D}\frac{k^2}{(k+p_3)^2(k-p_4)^2}=-\frac{s}{2}\,\mathrm{Bub}(s;D)\,. \tag{7}$$

3. (Bonus) Use the integration-by-parts identities

$$0=\int\frac{d^Dk}{(2\pi)^D}\frac{\partial}{\partial k^\mu}\left(\frac{k^\mu}{k^2(k+p_3)^2(k-p_4)^2}\right)\,,\quad 0=\int\frac{d^Dk}{(2\pi)^D}\frac{\partial}{\partial k^\mu}\left(\frac{k^\mu+p_3^\mu}{(k+p_3)^2(k-p_4)^2}\right)\,, \tag{8}$$

to show that the massless triangle integral is in fact reducible to the massless bubble

$$\widetilde{\mu}^{2\varepsilon}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2(k+p_3)^2(k-p_4)^2}=-\frac{2(D-3)}{s(D-4)}\,\mathrm{Bub}(s;D)\,. \tag{9}$$

4. (Bonus) Finally, employ one of the following two strategies to show eq. (4):
   a) Use Lorentz covariance to write down an ansatz for $\tilde{I}^{\rho\lambda}$ in terms of all possible rank-2 tensors and fix the coefficients by considering suitable contractions.
   b) As you already argued in the lecture, it follows from general considerations and Lorentz covariance that

$$\bar{u}(p_3)\,I^\mu\,u(p_4)=I(s)\,\bar{u}(p_3)\gamma^\mu u(p_4)\,. \tag{10}$$

   To compute the scalar function $I(s)$, construct a projector $P_\mu\propto\bar{u}(p_4)\gamma^\mu u(p_3)$ such that

$$\sum_{\text{spins}}P_\mu\,\bar{u}(p_3)\,I^\mu\,u(p_4)=I(s)\,. \tag{11}$$

With eq. (4), the leading virtual correction to the cross section was derived in the lecture to read

$$\sigma_V^{(1)}=\frac{\alpha_s}{2\pi}\,\sigma^{(0)}\left(\frac{s}{\mu^2}\right)^{-\varepsilon}C_F\left[-\frac{2}{\varepsilon^2}-\frac{3}{\varepsilon}-8+\frac{7\pi^2}{6}+\mathcal{O}(\varepsilon)\right]\,,\quad C_F=\frac{N_c^2-1}{2N_c}\,. \tag{12}$$

1.2 Real QCD corrections

The real corrections to the process $e^-e^+\to q\bar{q}$ at $\mathcal{O}(\alpha_s)$ are given by the emission of a real gluon g from the final state quark or antiquark,

$$e^-(p_1)+e^+(p_2)\to q(p_3)+\bar{q}(p_4)+g(k)\,. \tag{13}$$

1. Explain why the cross section of the real emission process, albeit the apparently different final state, should be added with the cross section for $e^-e^+\to q\bar{q}$ to define a sensible physical observable.

2. Draw the two diagrams and write down the corresponding amplitude $\mathcal{M}_R$ using Feynman rules in Feynman gauge.

[1] Hint: you can use a shift in the loop momentum $k\to k+p_4$ and a suitable ansatz for the resulting tensorial integral.

The differential cross section for the process may be split into a leptonic and a hadronic part,

$$\mathrm{d}\sigma_R^{(1)} = \frac{1}{2s}\,\mathrm{d}\Pi_3 \left(\frac{1}{4} \sum_{\substack{\text{spins}\\ \text{colours}}} |\mathcal{M}_R|^2 \right) = \frac{(4\pi)^3\alpha^2\alpha_s}{2s^3}\, Q_q^2\, L_{\mu\nu}\, X^{\mu\nu}\,, \tag{14}$$

where the leptonic tensor $L_{\mu\nu}$ reads

$$L_{\mu\nu} = \frac{1}{4}\sum_{\text{spins}} \bar{u}(p_2)\gamma_\mu u(p_1)\bar{u}(p_1)\gamma_\nu u(p_2)\,. \tag{15}$$

The integration over the D-dimensional three-particle phase space

$$\mathrm{d}\Pi_3 = \widetilde{\mu}^{4\varepsilon} \frac{\mathrm{d}^{D-1}p_3}{(2\pi)^{D-1}\,2E_{p_3}} \frac{\mathrm{d}^{D-1}p_4}{(2\pi)^{D-1}\,2E_{p_4}} \frac{\mathrm{d}^{D-1}k}{(2\pi)^{D-1}\,2E_k}\,(2\pi)^D\,\delta^{(D)}(p_1+p_2-p_3-p_4-k) \tag{16}$$

has been absorbed in the hadronic tensor $X^{\mu\nu}$, which contains all the dependence of the amplitude squared on the final state momenta.

3. The hadronic tensor is associated with the decay of the intermediate off-shell photon into the quark-antiquark pair and the gluon. Use this to show that[2]

$$X^{\mu\nu} = \left((p_1+p_2)^\mu(p_1+p_2)^\nu - s\,g^{\mu\nu}\right) X(s)\,, \quad \text{where} \quad X(s) = \frac{g_{\mu\nu}\,X^{\mu\nu}}{(1-D)s}\,. \tag{17}$$

Instead of working with scalar products of momenta, it is convenient to introduce new variables as

$$x_1 = 1 - \frac{(p_3+k)^2}{s} = 1 - \frac{(p_1+p_2-p_4)^2}{s} = \frac{2\,(p_1+p_2)\cdot p_4}{s}\,, \tag{18}$$

$$x_2 = 1 - \frac{(p_4+k)^2}{s} = 1 - \frac{(p_1+p_2-p_3)^2}{s} = \frac{2\,(p_1+p_2)\cdot p_3}{s}\,, \tag{19}$$

$$x_\gamma = 1 - \frac{(p_3+p_4)^2}{s} = 1 - \frac{(p_1+p_2-k)^2}{s} = \frac{2\,(p_1+p_2)\cdot k}{s}\,. \tag{20}$$

These are essentially the energies of the final state particles in the rest frame of the decaying photon. From momentum conservation, it follows that

$$x_1 + x_2 + x_\gamma = \frac{2}{s}\,(p_1+p_2)\cdot(p_3+p_4+k) = 2\,. \tag{21}$$

4. Use your expression for $\mathcal{M}_R$ from Feynman rules to extract $X^{\mu\nu}$ from eq. (14) and show that

$$g_{\mu\nu}\,X^{\mu\nu} = 2\,N_c\,C_F\,(2-D)\,\mathrm{d}\Pi_3\,\frac{2\,x_1^2 + 2\,x_2^2 + (D-4)\,x_\gamma^2}{(1-x_1)(1-x_2)} \tag{22}$$

[2] Note that the Ward identity for the photon holds no matter whether it is on-shell or not. For the proof, see Schwartz, *QFT and the SM* (2013), section 14.8.3.

If you wish to avoid the tedious part of the task of computing the huge trace, you may take for granted that

$$\mathrm{tr}\left(\not{p}_3\, S^{\mu\nu}\, \not{p}_4\, S_{\nu\mu}\right) = 2(D-2)\,\frac{2\,x_1^2 + 2\,x_2^2 + (D-4)\,x_\gamma^2}{(1-x_1)(1-x_2)}\,, \tag{23}$$

where

$$S^{\mu\nu} = \gamma^\nu \frac{\not{p}_3 + \not{k}}{(p_3+k)^2}\gamma^\mu - \gamma^\mu \frac{\not{p}_4 + \not{k}}{(p_4+k)^2}\gamma^\nu\,. \tag{24}$$

5. Demonstrate explicitly that the phase space integral can be simplified to

$$\mathrm{d}\Pi_3 = \left(\frac{s}{\mu^2}\right)^{-2\varepsilon} \frac{s\, e^{2\varepsilon\gamma_E}}{128\pi^3\,\Gamma(2-2\varepsilon)} \int_0^\infty \mathrm{d}x_1 \int_0^\infty \mathrm{d}x_2 \int_{r_-}^{r_+} \mathrm{d}x_\gamma\, \frac{\delta(2-x_1-x_2-x_\gamma)}{[(1-x_1)(1-x_2)(1-x_\gamma)]^\varepsilon}\,, \tag{25}$$

where $r_\pm = \sqrt{(x_1 \pm x_2)^2}$. Check that after eliminating x_γ by the constraint from the δ-function, the region to integrate x_1 and x_2 over is given by $\{0 \le x_1 \le 1, 0 \le x_2 \le 1\} \cap \{1 \le x_1 + x_2\}$.

6. (Bonus) Parametrize the remaining integration region as $x_1 = x$, $x_2 = 1 - x\,y$ with $0 \le x \le 1$, $0 \le y \le 1$ and perform the integrals to verify that

$$g_{\mu\nu}\, X^{\mu\nu} = -N_c\, C_F\, \frac{s}{8\pi^3}\,(1-2\varepsilon)(\varepsilon^2 - 2\varepsilon + 2)e^{2\varepsilon\gamma_E}\,\frac{\Gamma(-\varepsilon)^2\,\Gamma(2-\varepsilon)}{\Gamma(2-2\varepsilon)\,\Gamma(3-3\varepsilon)}\left(\frac{s}{\mu^2}\right)^{-2\varepsilon}. \tag{26}$$

7. Finally, put all the pieces together and check that the final result for the real correction to the cross section reads

$$\sigma_R^{(1)} = \frac{\alpha_s}{2\pi}\,\sigma^{(0)}\left(\frac{s}{\mu^2}\right)^{-\varepsilon} C_F\left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{57}{6} - \frac{7\pi^2}{6} + \mathcal{O}(\varepsilon)\right] \tag{27}$$

such that all IR divergences indeed cancel in the sum of real and virtual corrections,

$$\sigma^{(0)} + \sigma_V^{(1)} + \sigma_R^{(1)} \xrightarrow{\varepsilon\to 0\,,\,N_c=3} \frac{4\pi\alpha^2}{s} Q_q^2\left(1 + \frac{\alpha_s}{\pi}\right). \tag{28}$$

2 Universality of the eikonal current in QCD

2.1 Soft gluon emission from external gluons

Let's take any process producing two gluons,

$$X \to\; g(p_1, a_1) + g(p_2, a_2)\,, \tag{29}$$

where p_i and a_i denote momentum and colour, respectively. Assuming that X is a colour singlet state, the corresponding amplitude can be written as

$$\mathcal{M}_{gg} = \delta^{a_1 a_2}\mathcal{M}_X^{\mu\nu}(p_1,p_2)\,\epsilon_\mu(p_1)\,\epsilon_\nu(p_2)\,. \tag{30}$$

Consider now the real corrections to this process given by emission of an additional gluon from one of the two final state gluon legs,

$$X \to g(p_1, a_1) + g(p_2, a_2) + g(k, b)\,. \tag{31}$$

Show that in the soft limit, $k^\mu \to 0$, the amplitude $\mathcal{M}_{ggg}$ for the real corrections behaves as

$$\mathcal{M}_{ggg} \xrightarrow{k^\mu \to 0} i\, g_s\, f^{a_1 a_2 b} \left(\frac{p_1 \cdot \epsilon(k)}{p_1 \cdot k} - \frac{p_2 \cdot \epsilon(k)}{p_2 \cdot k} \right) \mathcal{M}_X^{\mu\nu}(p_1, p_2)\, \epsilon_\mu(p_1)\, \epsilon_\nu(p_2) \tag{32}$$

for physical gluons. In the case of a more complicated colour dependent $\mathcal{M}^{\mu\nu}(p_1, p_2)$ in eq. (30), a behaviour similar to eq. (32) can be shown on the level of individual colour-ordered partial amplitudes.

2.2 Soft gluon emission from external massive quarks

Consider now instead the production of a *massive* quark-antiquark pair from a colour singlet state X,

$$X \to q(p_1, i) + \bar{q}(p_2, j)\,, \tag{33}$$

with amplitude

$$\mathcal{M}_{q\bar{q}} = \delta^{ij}\, \bar{u}(p_1)\, \mathcal{M}_X(p_1, p_2)\, v(p_2)\,. \tag{34}$$

Show that the amplitude for the real corrections (emission of a real gluon $g(k, a)$ from the final state quark or antiquark) behaves in the soft limit as

$$\mathcal{M}_{gq\bar{q}} \xrightarrow{k^\mu \to 0} g_s\, T^a_{ij} \left(\frac{p_1 \cdot \epsilon(k)}{p_1 \cdot k} - \frac{p_2 \cdot \epsilon(k)}{p_2 \cdot k} \right) \bar{u}(p_1)\, \mathcal{M}_X(p_1, p_2)\, v(p_2)\,. \tag{35}$$