# Advanced Quantum Field Theory SS 2023

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**Sheet 6: YM Quantization and One Loop Divergences**

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## 1 Faddeev-Popov procedure

The Faddeev-Popov procedure allows for a consistent quantisation of gauge theories. Starting with the Yang-Mills action

$$S_{\mathrm{YM}} = \int d^4x\, \mathcal{L}_{\mathrm{YM}}, \tag{1}$$

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2} Tr[F^{\mu\nu}F_{\mu\nu}], \qquad F^{\mu\nu} = -ig\,(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]), \tag{2}$$

we fix the gauge through a functional constraint of the form

$$\delta\,(\mathcal{G}[A^\alpha]^a)\,\det\Big(\frac{\delta \mathcal{G}[A^\alpha]}{\delta\alpha}\Big), \tag{3}$$

and then integrate over the gauge transformation parameter function $\alpha$. The field $A^\alpha$ is the gauge-transformed field, according to the infinitesimal rule

$$(A^a_\mu)^\alpha = A^a_\mu + \frac{1}{g}(\partial_\mu \alpha^a + g f^{abc} A^b_\mu \alpha^c) \tag{4}$$

The path integral takes then the form,

$$\int [\mathcal{D}\alpha] \int [\mathcal{D}A]\; \delta\,(\mathcal{G}[A^\alpha]^a)\,\det\Big(\frac{\delta \mathcal{G}[A^\alpha]}{\delta\alpha}\Big) \mathrm{e}^{\{iS_{\mathrm{YM}}\}} \tag{5}$$

The next step is to perform the change of variables

$$A^a_\mu \longrightarrow \bar{A}^a_\mu \equiv A^\alpha_\mu, \tag{6}$$

which is effectively a gauge transformation. The action is gauge invariant by construction $S[A] = S[\bar{A}]$.

1. Verify that the integration measure is invariant as well, $\mathcal{D}A = \mathcal{D}\bar{A}$.

2. The extra functional determinant can be exponentiated and interpreted as ghosts particles interacting with the gauge field. Show that in the axial gauge, $r_\mu A^\mu = 0$, the functional determinant does not depend on the gauge field.

## 2 One loop calculations in QCD

In this exercise, we compute divergences of the QCD one loop Feynman integrals. More precisely, our goal is to compute all the pieces which are required for the famous QCD-$\beta$ function. To this end we must compute the divergent pieces of the gluon self-energy $-i\,\Sigma_g(q)$, the fermion self-energy $-i\,\Sigma_f(q)$ and the one loop correction to the gluon-quark vertex. Throughout the exercise sheet, we work in dimensional regularization and the Feynman gauge $\xi = 1$.

### 2.1 One loop corrections to the Fermion Self-Energy

**Step 1:** Draw the only diagram contributing to the fermion self-energy.
**Step 2:** Perform the color algebra and write down the loop integral in Feynman parameter representation.
**Step 3:** Evaluate the divergent part proportional to $\varepsilon^{-1}$ of the integral.

### 2.2 One loop corrections to the Gluon Self-Energy

To this end, we start with the gluon self-energy $-i\,\Sigma_g(q)$, where $q^\mu$ is the gluon momentum.
**Step 1:** Draw the 4 one-loop diagrams contributing to the gluon self-energy. Remember the ghost contributions.
**Step 2:** Classify the diagrams:

- There is one diagram which is 0 in dimensional regularization, which one? Why is it 0?
- There is one diagram which is familiar from the photon self-energy in QED, which one? Perform the color algebra and adjust the normalization. Write down the divergent piece.
- We compute the remaining two, in the next steps.

**Step 3:** Perform the color algebra in the remaining two diagrams and write down the integrals in Feynman parameter representation. Remember the symmetry factor $1/2$ for the gluon contribution and the factor $-1$ from the ghost contribution.
**Step 4:** Evaluate the integrals to order $\varepsilon^{-1}$ (you can use Mathematica for this, if you want).
**Step 5:** Put the pieces together, you should find that the gluon self-energy is transverse

$$-i\,\Sigma_g^{\mu\nu}(q) = -i\,q^2\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)\frac{\Pi(q^2)}{\epsilon}\,\delta_{ab}\,. \tag{7}$$

### 2.3 One loop corrections to the Gluon-Quark Vertex

**Step 1:** Draw the two diagrams contributing to the gluon-quark vertex correction.
**Step 2:** Perform the color algebra.
**Step 3:** Evaluate the UV divergent part of the integral. Because the integral has a superficially logarithmic divergence, you can neglect external momenta in the integrand and just keep the loop momentum dependence. Note the useful formula, see Eq. (B.49) in Schwartz for more information,

$$\left[\int\frac{d^D}{(2\pi)^D}\frac{1}{k^4}\right]_{\text{UV-div}} = \frac{i}{16\pi^2}\frac{1}{\varepsilon_{UV}}. \tag{8}$$

With this, you have computed all divergent pieces relevant for the calculation of the QCD beta function. We will look into this in the next exercise.