Advanced Quantum Field Theory SS 2023

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Sheet 06: Tree level amplitude for four-gluon scattering

The following problem contains a number of bonus questions, which you may attempt to solve at your own discretion.

Tree-level amplitude for the scattering of four gauge bosons

In this exercise, we compute the tree-level amplitude for the scattering of four gauge bosons in a non-abelian gauge theory based on the gauge group $SU(N)$. We take all particles as incoming

$$g_{\lambda_1,a_1}(p_1) \;+\; g_{\lambda_2,a_2}(p_2) \;+\; g_{\lambda_3,a_3}(p_3) \;+\; g_{\lambda_4,a_4}(p_4) \to 0\,, \tag{1}$$

where λ_i and a_i denote the helicity and colour of the corresponding gauge boson, respectively. Momentum conservation implies $p_1 + p_2 + p_3 + p_4 = 0$.

1. Draw the four Feynman diagrams that contribute to the amplitude at tree level.

Imagine, we would consider instead the gauge group $U(N) = SU(N) \times U(1)$. This means adding another gauge boson to the theory that is associated with the abelian gauge group $U(1)$, a "photon". Its generator is proportional to the identity, which is *not* traceless, as the constraint on the determinant for $SU(N)$ is lifted in $U(N)$. However, it commutes trivially with all $SU(N)$-generators, which implies that it does not couple directly to the other gauge bosons. Consequently, the amplitudes for the scattering of n gauge bosons are identical for $SU(N)$ and $U(N)$ Yang-Mills theory. At tree level, this remains true even if fermions are added to the theory, as they cannot appear in tree-level diagrams with only gauge bosons in the final and initial states. The inclusion of the photon will simplify the calculations at several points, as you will see soon.

2. Working now in the theory based on the gauge group $U(N)$, the Feynman rules for the gauge boson vertices remain unchanged, albeit colour indices might take on an extra value corresponding to the photon. As we just argued, this doesn't give an extra contribution to the amplitude as all structure constants f^{abc} involving the photon index vanish. Write down the amplitude explicitly, extract the colour factors and show that it can be decomposed into colour-ordered *partial amplitudes*

$$\mathcal{M}^{\lambda_1,\lambda_2,\lambda_3,\lambda_4}_{a_1,a_2,a_3,a_4}(p_1,p_2,p_3,p_4) = 4\sum_{\sigma\in S_3} \mathrm{tr}\left(T^{a_1}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}\right)\mathcal{A}\left[1,\sigma(2),\sigma(3),\sigma(4)\right]\,, \tag{2}$$

where the sum runs over all permutations of $\{2,3,4\}$ and T^{a_i} denotes the generator a_i in the fundamental representation. You may find it helpful to write

$$f^{abc} = -2\,i\,\mathrm{tr}\left(\left[T^a,T^b\right]T^c\right) \tag{3}$$

and to make use of the Fierz completeness relation for $U(N)$:

$$T^a_{ij}\,T^a_{kl} = \frac{1}{2}\,\delta_{il}\,\delta_{jk}\,. \tag{4}$$

3. How does the Fierz identity look like in the case of $SU(N)$, i.e. without the contribution from the photon generator? Show that this leads to the same colour structures as above.

The partial amplitudes have a number of useful properties. For once, they are all gauge-invariant on their own.

4. (Bonus) Check the partial orthogonality condition for your choice of $U(N)$ or $SU(N)$ and any permutation $\sigma \in S_3$

$$\sum_{a_1,a_2,a_3,a_4} \mathrm{tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right) \mathrm{tr}\left(T^{a_1}T^{a_{\sigma(2)}}T^{a_{\sigma(3)}}T^{a_{\sigma(4)}}\right)^* = \frac{N^2}{8}(N^2-\alpha)\left[\prod_{i=2}^{4}\delta_{i,\sigma(i)} + \mathcal{O}(N^{-2})\right], \quad (5)$$

   where $\alpha = 0$ for $U(N)$ and $\alpha = 1$ for $SU(N)$. The colour traces are hence linearly independent at leading order in the expansion and as gauge invariance must hold order-by-order in $1/N$, this implies the gauge invariance of the partial amplitudes.

Further, they satisfy a number of different relations that we will not prove here:

$$\text{(a)}\quad \textit{Cyclicity}: \quad \mathcal{A}[1,2,3,4] = \mathcal{A}[4,1,2,3] = \mathcal{A}[3,4,1,2] = \mathcal{A}[2,3,4,1]\ , \quad (6)$$

$$\text{(b)}\quad \textit{Reflection symmetry}: \quad \mathcal{A}[1,2,3,4] = \mathcal{A}[4,3,2,1]\ , \quad (7)$$

$$\text{(c)}\quad \textit{Photon decoupling}: \quad \mathcal{A}[1,2,3,4] + \mathcal{A}[2,1,3,4] + \mathcal{A}[2,3,1,4] = 0\,. \quad (8)$$

One way to show the last identity is by choosing one of the external gauge bosons to be the photon and imposing that the amplitude vanishes, hence the name. These identities are a consequence of the fact that the four-colour traces are an *overcomplete* basis to describe the scattering of four gauge bosons. To see this, notice that there are six different colour traces involving four generators, but the Feynman rules produce only three different colour structures in the form of products of two structure constants. Further, only two of them are actually independent because of the Jacobi identity.

5. Show that the above relations reduce the number of independent partial amplitudes to two, for which we can choose $\mathcal{A}[1,2,3,4]$ and $\mathcal{A}[1,2,4,3]$.

It turns out that there exist even more linear relations. The so-called *BCJ relation* states

$$(p_1+p_4)^2\mathcal{A}[1,2,3,4] + (p_1+p_3)^2\mathcal{A}[1,2,4,3] = 0 \quad (9)$$

and therefore we end up with only a single independent partial amplitude, say $\mathcal{A}[1,2,3,4]$, which we have to calculate explicitly.

6. Take the expression from Feynman rules for the whole amplitude and extract the piece corresponding to $\mathcal{A}[1,2,3,4]$. Which Feynman diagrams contribute to this partial amplitude?

To perform the computation, we will employ spinor-helicity formalism and write $\mathcal{A}\left[1^{\lambda_1},2^{\lambda_2},3^{\lambda_3},4^{\lambda_4}\right]$ for the individual helicity amplitudes.

7. Imagine you have an expression for $\mathcal{A}\left[1^{\lambda_1},2^{\lambda_2},3^{\lambda_3},4^{\lambda_4}\right]$ in terms of spinor products. Explain why the amplitude with flipped helicities $\mathcal{A}\left[1^{-\lambda_1},2^{-\lambda_2},3^{-\lambda_3},4^{-\lambda_4}\right]$, can be obtained by swapping angle and square brackets as follows $\langle ij\rangle \leftrightarrow [ji]$ (notice the different order).

8. How many helicity configurations are there in total? How many do you have to consider explicitly?

9. Prove that the partial amplitudes with all plus helicities or all plus and one minus helicities are zero
$$\mathcal{A}\left[1^+,2^+,3^+,4^+\right]=0\,,\qquad \mathcal{A}\left[1^+,2^+,3^+,4^-\right]=0\,.$$
To do that, use the fact that the reference vector for each external gauge boson can be chosen freely in every individual helicity amplitude but the choice has to be consistent across all diagrams. This implies that the only non-zero helicity amplitudes are the ones with at least two minus helicities, usually referred to as *maximally helicity violating (MHV)*.

10. The relations among partial amplitudes from equations (6), (7) and (8) have to be satisfied also at the level of individual helicity amplitudes. Using this and crossing symmetry for the external gauge bosons, show that there is only one independent MHV amplitude, which we can choose as $\mathcal{A}[1^+,2^+,3^-,4^-]$. To relate amplitudes with adjacent and non-adjacent equal helicities, you should prove that
$$\mathcal{A}\left[1^+,2^-,3^+,4^-\right]=-\mathcal{A}\left[1^+,3^+,2^-,4^-\right]-\mathcal{A}\left[1^+,3^+,4^-,2^-\right]\,. \tag{10}$$

11. Finally, perform an explicit computation for the only independent non-zero helicity amplitude and derive
$$\mathcal{A}\left[1^+,2^+,3^-,4^-\right]=g^2\,\frac{\langle 12\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 41\rangle}\,. \tag{11}$$

12. (Bonus) Using equation (10), show that
$$\mathcal{A}\left[1^+,2^-,3^+,4^-\right]=g^2\,\frac{\langle 13\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 41\rangle}\,. \tag{12}$$

With the derivation of equations (11) and (12), you have proven two important special cases of the famous *Parke-Taylor formula*, which provides a closed formula for MHV amplitudes for the scattering of n gauge bosons at tree level as follows
$$\mathcal{A}\left[1^-,2^-,...,i^+,...,j^+,...,(n-1)^-,n^-\right]\propto\frac{\langle ij\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,...\,\langle (n-1)n\rangle\,\langle n1\rangle}\,. \tag{13}$$

This formula can be proven using on-shell methods, for example by the so-called BCFW recursion relation [1, 2]. You are invited to try to think how complicated it would be to derive this result by direct calculation by Feynman diagrams for $n\geq 6$!

References

[1] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B **715** (2005), 499-522 doi:10.1016/j.nuclphysb.2005.02.030 [arXiv:hep-th/0412308 [hep-th]]. 3

[2] R. Britto, F. Cachazo, B. Feng and E. Witten, Phys. Rev. Lett. **94** (2005), 181602 doi:10.1103/PhysRevLett.94.181602 [arXiv:hep-th/0501052 [hep-th]]. 3