Advanced Quantum Field Theory SS 2023

Prof. Lorenzo Tancredi (TUM)

https://www.groups.ph.tum.de/ttpmath/teaching/ss-2023/

Sheet 05: Naive quantization of gauge theories

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1 Ghosts, unitarity and gauge invariance

For simplicity, in this exercise we work with massless fermions. In sheet 2 we computed amplitudes for di-photon production in QED at tree level

$$e^{-}(p_1) + e^{+}(p_2) \longrightarrow \gamma(p_3) + \gamma(p_4)$$

Gauge invariance implies that if a photon polarisation vector is replaced by its momentum, then the amplitude vanishes

$$\mathcal{M} = \epsilon(p)^{\mu}\mathcal{M}_{\mu} \longrightarrow p^{\mu}\mathcal{M}_{\mu} = 0, \tag{1}$$

an equation known as Ward Identity.

1.1 The QED case

1. Write again the expression for the two diagrams contributing to this process and check that the Ward identity is satisfied (it is sufficient to check one of the two). Note that there is no need to fix helicities: solve the exercise for general spinors in the usual u, $\bar{u}$ notation.

1.2 The non-abelian case (QCD)

Consider now the non-abelian generalization, based on the group SU(N). For definiteness we call the fermions quarks, and the gauge bosons gluons.
As you have seen in the lecture, the *classical* Yang-Mills Lagrangian reads

$$\begin{aligned}\mathcal{L}^{\text{kin}} &= -\frac{1}{4}F^{a}_{\mu\nu}F^{\mu\nu,a} + \bar{\psi}(i\not{D})\psi \\ &= \mathcal{L}_{kin} + \mathcal{L}^{\text{int}}_{q\bar{q}g} + \mathcal{L}^{\text{int}}_{3g} + \mathcal{L}^{\text{int}}_{4g},\end{aligned} \tag{2}$$

where

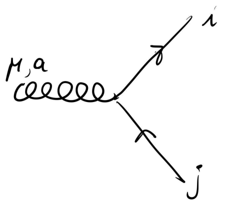
$$\begin{aligned}\mathcal{L}^{\text{int}}_{q\bar{q}g} &= gA^{a}_{\mu}\bar{\psi}_i\gamma^{\mu}T^{a}_{ij}\psi_j, \\ \mathcal{L}^{\text{int}}_{3g} &= -gf^{abc}\left(\partial_{\mu}A^{a}_{\nu}\right)A^{\mu,b}A^{\nu,c}, \\ \mathcal{L}^{\text{int}}_{4g} &= -\frac{1}{4}g^{2}f^{eab}f^{ecd}A^{a}_{\mu}A^{b}_{\nu}A^{\mu,c}A^{\nu,d}.\end{aligned} \tag{3}$$

To be able to formally quantize this Lagrangian, one needs to add a gauge fixing term, similarly to QED, and also a new unphysical field, which goes under the name of *ghost*. In this exercise we will see the necessity of this field, by naive direct calculation.

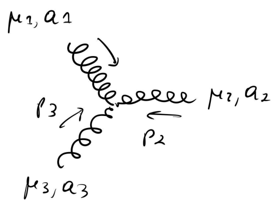
1. The gluon propagator can be obtained adding a gauge fixing terms $\mathcal{L}_{fix} = -\frac{1}{2\xi}(\partial^{\mu}A^{a}_{\mu})^2$, and similarly to QED reads:

$$\Pi^{ab}_{\mu\nu}(p) = i\,\delta^{ab}\left(-g_{\mu\nu} + (1-\xi)\frac{p^{\mu}p^{\nu}}{p^2}\right)$$

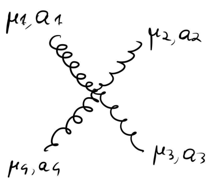
2. Derive the *naive* Feynman rules for the interaction vertices that stem from the lagrangian (3). You should find



$$= ig_s T^a_{ij} \gamma^{\mu_3} \,, \tag{4}$$



$$= g_s f^{a_1 a_2 a_3} \left[g^{\mu_1 \mu_2} (p_2 - p_1)^{\mu_3} + \text{cycl}(123) \right] \,, \tag{5}$$



$$= -ig_s^2 \Big\{ f^{a_1 a_2 b} f^{a_3 a_4 b} (g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} - g^{\mu_1 \mu_4} g^{\mu_2 \mu_3}) + \text{cycl}(234) \Big\} . \tag{6}$$

With this, consider the production of two gluons in quark anti-quark annihilation, where the quark fields are massless

$$q\,(p_1) + \bar{q}\,(p_2) \longrightarrow g\,(p_3) + g\,(p_4) \tag{7}$$

and from now on work in the Feynman gauge $\xi = 1$.

1. Draw the diagrams that contribute to the process in (7).

2. Consider only the two diagrams analogous to the abelian case, i.e. ignoring the gluon self-interaction term. Check that the Ward identities are *not satisfied*. For definiteness, do this for the second gluon, i.e. $\epsilon_4^\mu(p_4) \to p_4^\mu$.

3. Write the expression for the third (non-abelian) diagram and show that this fixes the problem, assuming that the *the other gluon is on-shell*, i.e. $\epsilon_3(p_3) \cdot p_3 = 0$.

1.3 Ghosts and unitarity

Assuming $\epsilon(p) \cdot p = 0$ was not necessary to prove the Ward Identity in QED.

1. Argue why in QED, when taking the modulus squared of the amplitude, we can safely replace the photon polarisation sum with $-g_{\mu\nu}$ (which includes non physical polarisations). Why can we not do the same in the non abelian case?

2. Consider the explicit representation for the outgoing longitudinal polarisation vectors

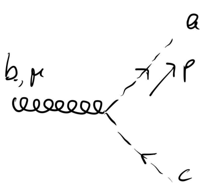
$$\epsilon_\mu^+(k) = \frac{k^\mu}{\sqrt{2}|\vec{k}|}\,, \qquad \epsilon_\mu^-(k) = \frac{\tilde{k}^\mu}{\sqrt{2}|\vec{k}|} \tag{8}$$

where $k^\mu = (k_0, \vec{k})$, and $\tilde{k}^\mu = (k_0, -\vec{k})$. Show that the amplitude for the case $\epsilon_3^+(p_3)$ and $\epsilon_4^+(p_4)$ is zero whereas the amplitude for the case $\epsilon_3^-(p_3)$ and $\epsilon_4^+(p_4)$ is non zero.

3. We add now to the Lagrangian a new complex field c^a which is a scalar under Poincare, transforms under the adjoint representation of SU(N) and has *(non-physical)* fermionic spin-statistics:

$$\mathcal{L}_{gh} = \bar{c}^a(-\partial^\mu D_\mu^{a,b})c^b = \mathcal{L}_{gh}^{kin} + g\, f^{abc}\partial_\mu \bar{c}^a A_\mu^b c^c.$$

Derive the Feynman rules for the interaction of this new field with the gluon, you should find:



$$= -\, g f^{abc} p^\mu \tag{9}$$

4. Compute now the amplitude for the production of two ghosts in quark-antiquark annihilation

$$q(p_1) + \bar{q}(p_2) \to c(p_3) + \bar{c}(p_4)$$

and convince your self that this contribution exactly matches the one from the longitudinal polarizations of the external gluons.

5. Consider now the one loop forward amplitude for

$$q(p_1) + \bar{q}(p_2) \to q(p_1) + \bar{q}(p_2)$$

allowing both ghosts and gluons to propagate in the loop. Convince yourself that the ghost contribution exactly cancels the one from the longitudinally polarized gluons, as required from the optical theorem.