Advanced Quantum Field Theory SS 2023

Lecturer: Prof. Lorenzo Tancredi
Assistants: Philipp Alexander Kreer, Cesare Mella, Nikolaos Syrrakos, Fabian Wagner
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Sheet 04: Yang-Mills classical field theory

Problem 1 - Additional term in the non-abelian Lagrangian

For this exercise we could consider a non-abelian gauge field theory with an arbitrary gauge group. For definiteness, let us take the explicit case of $SU(N)$. The Lagrangian density is given by

$$\mathcal{L}_A = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu,a} = -\frac{1}{2}\text{Tr}\left[F_{\mu\nu}F^{\mu\nu}\right]$$

with $F^b_{\mu\nu} = \partial_\mu A^b_\nu - \partial_\nu A^b_\mu - gf^{bcd}A^c_\mu A^d_\nu$. f^{abc} are the structure constants, g is a constant and represents the coupling. The indices a, b, c are in the adjoint representation, such that $F^{\mu\nu,a}$ in general transforms non-trivially under gauge transformations.
At first sight, it appears possible to add to the standard gauge field Lagrangian the term

$$\delta\mathcal{L} = \epsilon_{\mu\nu\rho\sigma}\text{Tr}\left[F^{\mu\nu}F^{\rho\sigma}\right] .$$

1. Discuss how you expect this term to change under a parity transformation.
2. Show that this term is, in fact, a total derivative $\delta\mathcal{L} = \partial_\mu J^\mu$.
3. Find the expression for the vector current J^μ and explain why such terms do not change the equations of motion.

Problem 2 - Equations of motions in non-abelian field theory

Consider a non-abelian gauge theory where gauge fields couple to scalar fields in the fundamental representation of the gauge group

$$\mathcal{L} = \mathcal{L}_A + (D_\mu\phi)^\dagger(D^\mu\phi) - m^2\phi^\dagger\phi - \lambda\left(\phi^\dagger\phi\right)^2 ,$$

where $\mathcal{L}_A$ is the standard Lagrangian for the gauge fields given in the previous exercise.

1. Show that the equations of motion for the gauge field can be written in a form
$$(D_\mu F^{\mu\nu})^a = gJ^a_\nu \tag{1}$$
and find the current J^a_ν. Note that in Eq.(1) the covariant derivative acts on $F^{\mu\nu}$ which has values in the Lie Algebra and transforms in the adjoint representation! The covariant derivative is given by
$$D^{ab}_\mu = \partial_\mu - igf^{abc}A^c_\mu.$$
2. Show that $D_\mu D_\nu F^{\mu\nu} = 0$. Use this result to write down the conservation equation for the current J^a_μ.
3. Find the equations of motion for the scalar field ϕ.