Sheet 03: Group Theory

---

1 su(N) and the anomaly coefficients

We refer to the $\mathbf{su}(N)$ algebra generators in a given representation R as T_R^a, $a = 1, ..., d(\mathbf{su}(N))$, with $d(\mathbf{su}(N))$ the dimension of the algebra. If no subscript is present, then the fundamental representation is assumed. The totally antisymmetric invariant d^{abc} is defined as

$$d^{abc} = 2tr\left[T^a\{T^bT^c\}\right]. \tag{1}$$

The generalisation to an arbitrary representation R is given by

$$tr\left[T_R^a\{T_R^bT_R^c\}\right] = \frac{1}{2}A(R)d^{abc} = A(R)tr\left[T^a\{T^bT^c\}\right], \tag{2}$$

where the constant $A(R)$ is called anomaly coefficient. In the fundamental representation, $A(R) = 1$.

1. Consider a representation R and the complex conjugated one, $\bar{R}$. Show that $A(R) = -A(\bar{R})$. Argue that for a real representation $A(R) = 0$. (It is useful to first prove that $A(R)d^{abc}$ is real.)
2. Any reducible representation of a Lie algebra can be decomposed as the direct sum of irreducible representations. Show that if
$$R = R_1 \oplus R_2$$
then
$$A(R) = A(R_1 \oplus R_2) = A(R_1) + A(R_2). \tag{3}$$
Remember that the direct sum of two matrices A and B is the matrix C in block diagonal form, with A and B being the two blocks.
3. Show the analogous formula for tensor product of representations
$$A(R_1 \otimes R_2) = A(R_1)\,d(R_2) + d(R_1)\,A(R_2), \tag{4}$$
where $d(R)$ is the dimension of the corresponding representation. Remember that the tensor product of two matrices, A and B, of dimension m and n respectively, is defined as the matrix C with entries $C_{ij,lk} = A_{il}B_{jk}$. First prove that
$$(A \otimes B)(A' \otimes B') = (AA') \otimes (BB') \tag{5}$$
$\forall A, A'$ $m \times m$ matrices and B, B' $n \times n$ matrices. Then argue that
$$tr(A \otimes B) = tr(A)tr(B). \tag{6}$$
4. What is $A(\mathbf{10})$ for $SU(4)$? Use the fact that $\mathbf{4} \otimes \mathbf{4} = \mathbf{6} \oplus \mathbf{10}$ and that $\mathbf{6}$ is a real representation.

2 The coset space

Given a group G and a subgroup H of G, the coset space G/H is defined as follows. We define an equivalence relation $\sim$ among elements of the group as

$$g \sim g' \quad \text{if} \quad \exists\, h \in H \quad \text{s.t.}\ g' = gh, \ \ g, g' \in G. \tag{7}$$

The set of equivalence classes is called the (left) coset space of G by H, and indicated as G/H.

Consider now the group $SO(3)$ of rotation in 3-dimension and the subgroup given by rotation by the z-axis

$$SO(2) \simeq SO(2)_z \equiv \{R_z \in SO(3)\} \subset SO(3), \tag{8}$$

The exercise consists in proving that $SO(3)/SO(2)_z$ is in one-to-one correspondence with S^2, the two dimensional sphere.

1. Given that a rotation can always be written as the composition of a rotation by the z-axis and a rotation by an axis in the xy-plane, argue that the coset space can be built using as representatives for the equivalence classes all possible rotation by an axis in the xy-plane of an angle θ.

2. Define a map from $SO(3)/SO(2)_z$ to S^2. Hint: the unit vector $\hat{e}_z$ is invariant under the action of $SO(2)_z$.

3 The trace scalar product

It can be proved that a Lie algebra is compact if and only if there exists a positive-definite scalar product, which is invariant under the adjoint action of the group. Namely, the following conditions are true

$$(A, B) = (Ad(g)A, Ad(g)B) \ \ \forall g \in G, \tag{9}$$

and

$$(A, A) \geq 0. \tag{10}$$

For a matrix group, the scalar product is the trace

$$(A, B) = -Tr(A, B). \tag{11}$$

1. For a matrix group, prove that (11) is invariant under the action of the adjoint representation. The non trivial part of the theorem is proving the positive definiteness, which we will not do here.

2. Verify that (10) holds for $su(2)$ but not for $gl(2, C)$.