Advanced Quantum Field Theory SS 2023

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https://www.groups.ph.tum.de/ttpmath/teaching/ss-2023/

Sheet 2: Spinor-Helicity Formalism

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1 Explicit Representations

Remember

$$(\sigma^\mu)_{a\dot{b}} = \left(1, \sigma^i\right)_{a\dot{b}}, \quad (\bar{\sigma}^\mu)^{\dot{a}b} = \left(1, -\sigma^i\right)^{\dot{a}b},$$

with Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Furthermore, the 2 component Levi-Civita symbol is defined as

$$\varepsilon^{ab} = \varepsilon^{\dot{a}\dot{b}} = -\varepsilon_{ab} = -\varepsilon_{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

1.1 Explicit Spinor Representations

Consider the momentum vector,

$$p^\mu = (E, E\sin\theta\cos\phi, E\sin\theta\sin\phi, E\cos\theta)$$

Step 1: Express p_{ab} and $p^{\dot{a}b}$ in terms of $E, \sin\frac{\theta}{2}, \cos\frac{\theta}{2}$ and $e^{\pm i\phi}$.

Step 2: Show that the helicity spinor $|p\rangle^{\dot{a}} = \sqrt{2E}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}$ solves the massless Weyl equation.

Step 3: Find expressions for the spinors $\langle p|_{\dot{a}}, |p]_a$, and $[p|^a$ and check that they satisfy $p_{a\dot{b}} = |p]_a \langle p|_{\dot{b}}$ and $p^{\dot{a}b} = |p\rangle^{\dot{a}} [p|^b$.

1.2 Explicit Representation Polarization Vector

In this exercise, we establish the connection between the polarization vectors

$$\epsilon_-^\mu(p;q) = \frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle qp\rangle} \tag{1}$$

and the more familiar polarization vectors

$$\tilde{\epsilon}_-^\mu(p) = \frac{e^{-i\phi}}{\sqrt{2}}(0, \cos\theta\cos\phi + i\sin\phi, \cos\theta\sin\phi - i\cos\phi, -\sin\theta).$$

Note that for $\theta = \phi = 0$, we have $\tilde{\epsilon}_-^\mu(p) = \frac{1}{\sqrt{2}}(0, 1, -i, 0)$

Step 1: Since $\tilde{\epsilon}_-^\mu(p)$ is null, $(\tilde{\epsilon}_-^\mu(p))_{a\dot{b}} = (\sigma_\mu)_{a\dot{b}}\, \tilde{\epsilon}_-^\mu(p)$ can be written as a product of a square and an angle spinor. To see this specifically, first calculate $(\tilde{\epsilon}_-^\mu(p))_{a\dot{b}}$ and then find an angle spinor $\langle r|$ such that $(\tilde{\epsilon}_-^\mu(p))_{a\dot{b}} = |p]_a \langle r|_{\dot{b}}$. Verify that $\langle pr\rangle = \sqrt{2}$.

Step 2: Next, show that it follows from Eq. (1) that $(\epsilon_-(p;q))_{a\dot{b}} = \frac{\sqrt{2}}{\langle qp\rangle} |p]\, \langle q|$.

Step 3: Now suppose there is a constant c_- such that $\epsilon_-^\mu(p;q) = \tilde{\epsilon}_-^\mu(p) + c_- p^\mu$. Show that this relation requires $\langle pr\rangle = \sqrt{2}$ and then show that $c_- = \langle rq\rangle/\langle pq\rangle$.

1.3 Spinor identity

Prove the Fierz and reversal identity from the lecture.
Fierz identity:

$$[p\gamma^\mu q\rangle\, [k\gamma_\mu l\rangle = 2[pk]\langle lq\rangle$$
$$\langle p\gamma^\mu q]\, \langle k\gamma_\mu l] = 2\langle pk\rangle[lq]$$

Reversal identity:

$$\langle p\gamma^{\mu_1} \dots \gamma^{\mu_{2n}} q\rangle = -\, \langle q\gamma^{\mu_{2n}} \dots \gamma^{\mu_n} p\rangle$$
$$\langle p\gamma^{\mu_1} \dots \gamma^{\mu_{2n+1}} q] = [q\gamma^{\mu_n} \dots \gamma^{\mu_{2n+1}} p\rangle$$

2 QED Compton Scattering

In this exercise, we compute the tree level amplitude for electron-positron annihilation $\mathcal{A}(e^+e^- \to \gamma\gamma)$ using spinor-helicity formalism. We assume that the fermions are massless and take all particles as incoming

$$e^-(p_1) + e^+(p_2) + \gamma(p_3) + \gamma(p_4) \longrightarrow 0.$$

Step 1: What are the independent helicity configurations?
Step 2: Draw the two diagrams of this process.
Step 3: Insert the QED Feynman rules and perform the algebra using spinor-helicty formalism.
Step 4: Use the result from above to check that $\sum_{spins} |\mathcal{A}(spins)|^2$ gives the known result obtained from trace formalism à la Peskin & Schroeder, see Eq. (5.87).