Advanced Quantum Field Theory SS 2023

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https://www.groups.ph.tum.de/ttpmath/teaching/ss-2023/

Sheet 12: Anomalies and θ vacuum

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1 Problem 1: The Chiral Anomaly in the Schwinger model

We consider the so-called Schwinger model in $D = 2$ dimensions,

$$\mathcal{L} = -\frac{1}{4e_0^2} F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\not{D}\psi, \tag{1}$$

where spinors are two dimensional and gamma matrices are taken as

$$\gamma_0 = \sigma_2, \quad \gamma_1 = -i\sigma_1, \quad \gamma_5 = -\sigma_3. \tag{2}$$

QED in $D = 2$ dimensions is confining[1]: even for small values of the coupling, the theory is non-perturbative at large enough distances. If we want to apply perturbation theory we need to define the theory on an interval

$$-\frac{L}{2} < x < \frac{L}{2}, \tag{3}$$

and we assume $e \cdot L \ll 1$. On this interval, we use symmetric and anti-symmetric boundary condition for the field A_μ and ψ respectively.

We define the "left-handed" and "right-handed" spinors as

$$\psi_L = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix} \qquad \psi_R = \begin{pmatrix} 0 \\ \psi_2 \end{pmatrix}, \tag{4}$$

such that,

$$\gamma_5\,\psi_L = -\psi_L, \quad \gamma_5\,\psi_R = \psi_R. \tag{5}$$

It can be shown that is always possible to find a gauge transformation such that A_1 is constant and can be mapped into the interval $[0, \frac{2\pi}{L}]$. Moreover, at leading order in the Coulomb potential, we assume $A_0 = 0$. We will now proceed with the quantum mechanics of the system assuming that A_1 is an external constant field, which we adiabatically tune in the above interval. Later in the second exercise, we will quantize the gauge field on top of the fermionic system.

1. Verify that $\psi \longrightarrow e^{i\alpha}\psi$ and $\psi \longrightarrow e^{i\alpha\gamma_5}\psi$ are symmetries. The associated conserved currents are the vector, j_μ, and axial-vector, j^5_μ, currents. Show that you can split them as the sum of a left and a right component. What are the classically conserved charges for the two current, Q, Q_5? Split them in terms of a left, Q_L, and a right, Q_R, component.

2. Write down the Dirac equation for the ψ field and show that it decouples for left and right components.

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[1] The potential grows as $V(x) \approx e^2|x|$

3. Assuming $\psi \approx e^{-iE_k t}\psi_k(x)$, determine the energy level for the left and right components. You should find that

$$E_k^L = -\left(k+\frac{1}{2}\right)\frac{2\pi}{L} + A_1,$$
$$E_k^R = \left(k+\frac{1}{2}\right)\frac{2\pi}{L} - A_1, \tag{6}$$

for $k = 0, \pm 1, \pm 2, etc.$ Energy levels as a function of A_1 are displayed in fig.1.

4. Representing with $|1_{L/R}, k\rangle$ a particle in the energy level k, and with $|0_{L/R}, k\rangle$ the absence of a particle in that state, fill the Dirac sea at $A_1 = 0$ with the correct occupation number and build the vacuum state, $|\text{vac}\rangle$.

5. At this point, if we tried to compute the charges using canonical quantisation, we would get an infinite result. We need a UV regularisation: one is provided by the so-called split regularization for the vector and axial-vector currents

$$j^\mu_{reg} = \bar{\psi}(t, x+\epsilon)\gamma^\mu\psi(t,x)\exp\left(i\int_x^{x+\epsilon} A_1 dx\right),$$
$$j^{5,\mu}_{reg} = \bar{\psi}(t, x+\epsilon)\gamma^\mu\gamma^5\psi(t,x)\exp\left(i\int_x^{x+\epsilon} A_1 dx\right). \tag{7}$$

Show that definitions (7) are gauge invariant quantities.

6. With the energy dependence found in the previous point and the anti-symmetric boundary condition, we can expand the Left and Right field in terms of the annihilation operators as

$$\psi_L = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{L}} e^{i(k+\frac{1}{2})\frac{2\pi}{L}x - iE_k^L t}\ \hat{a}_{k,L},$$
$$\psi_R = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{L}} e^{i(k+\frac{1}{2})\frac{2\pi}{L}x - iE_k^R t}\ \hat{a}_{k,R}. \tag{8}$$

Compute the charges with the regularised current and evaluate them on the vacuum state previously determined. You will find that

$$Q_L = -Q_R = \frac{e^{i\epsilon A_1}}{2i\sin\left(\epsilon\pi/L\right)}. \tag{9}$$

Expand in ϵ the result up to the finite part.

7. Suppose A_1 is slowly varying. What can you conclude about Q and Q_A at the quantum level? Show that this is compatible with $\partial_\mu j^\mu_{5,\epsilon} \approx \epsilon^{\mu\nu}F_{\mu\nu}$. Explain the meaning of this equation in view of the results of point 1. This result is referred to as *chiral anomaly*, due to the anomalous non-conservation of the axial current due to quantum corrections.

8. How would you modify eq.(7) in the axial-vector current to get a conserved charge?

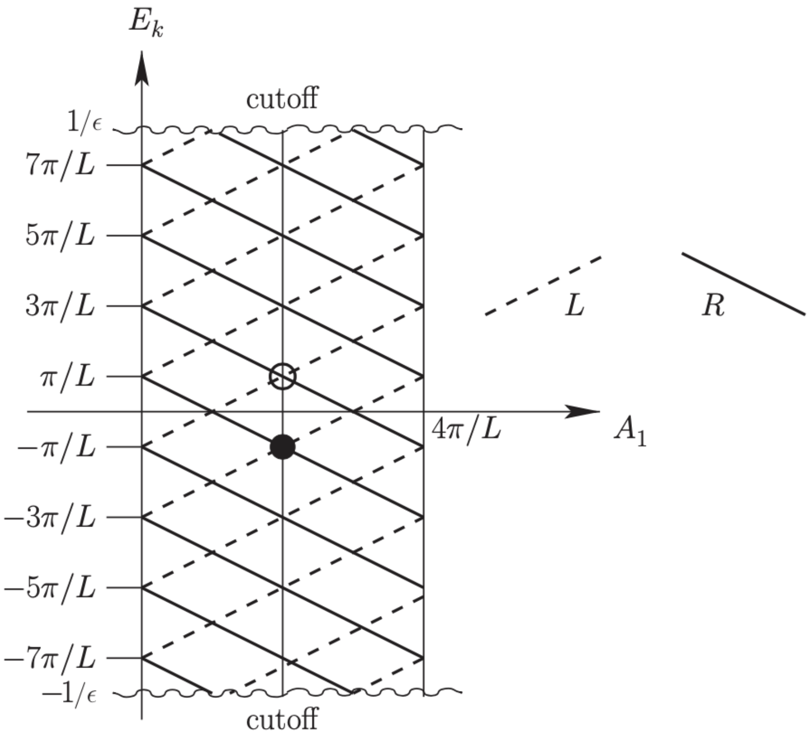

Figure 1: Fermionic energy levels.

2 Problem 2: the θ vacuum

We have seen how to introduce a UV regularization to treat the so-called "chiral anomaly" in the Schwinger model. In this problem we will have a closer look at the vacuum of the theory, computing the energy of the Dirac sea. We start writing down the fermion part of the Hamiltonian in the split-regularization as

$$H_\epsilon = \int_{-L/2}^{L/2} dx\, \psi(t, x+\epsilon)\sigma_3 \left(i\frac{\partial}{\partial x} + A_1 \right) \psi(t,x), \tag{10}$$

and the energy level of the "left-handed" and "right-handed" component of the Dirac sea look like

$$E_L = \sum_{k=0}^{\infty} E_{k(L)} \exp\left(i\epsilon E_{k(L)} \right), \qquad E_R = \sum_{k=-1}^{-\infty} E_{k(R)} \exp\left(i\epsilon E_{k(R)} \right), \tag{11}$$

with $0 \leq A_1 \leq 2\pi/L$.

1. What is the physical meaning of Eq.(11)? And how are the energy levels connected to the expression of the regularized "left-handed" and "right-handed" charges Q_L and Q_R?

2. Show that, by expanding in ϵ, the total energy of the sea is

$$E_{sea} = E_L + E_R = \frac{L}{2\pi} \left(A_1^2 - \frac{\pi^2}{L^2} \right) + \text{constant independent of } A_1. \tag{12}$$

3. It follows from the previous item that the energy of the Dirac sea quadratically depends on A_1; as the result fluctuations of A_1 in the vacuum can be described by an effective Lagrangian

$$\mathcal{L} = \frac{L}{e_0^2}(\partial_0 A_1)^2 - \frac{L}{2\pi}A_1^2. \tag{13}$$

Use the analogy with the quantum oscillator and explain how to quantize the quantum system described by the above Lagrangian. Determine the Hamiltonian and find the time dependence of the operators A_1. Find the ground state wave function of the quantum system described by the above Lagrangian $\psi_0(A_1)$ and explain its physical meaning.

4. Our treatment of the Schwinger model was based on the assumption that we can find fermion energy levels considering the field A_1 to be time-independent. Given the results obtained in the previous item and the energy levels of electrons in the Dirac sea, can you justify this approximation? You should look at the characteristic frequencies of the gauge field compared to those of the fermion sector. Remember that we had to assume that $e_0 L \ll 1$.

5. We can now build up the vacuum wave function in the vicinity of $A_1 = 0$, in the form

$$\psi_{vac} = \psi_{ferm.vac.}\psi_0(A_1), \tag{14}$$

with $\psi_{ferm.vac.}$ being the fermionic vacuum state found in the previous exercise. Show that this form of the wave function is not invariant under large gauge transformations, i.e. $A_1 \longrightarrow A_1 + 2\pi k/L$ with $k = \pm 1, \pm 2$ ...

6. Show that for every $n = 0, \pm 1, \pm 2...$, such that $A_1 \approx 2\pi n/L$, the Hilbert space splits into distinct sectors corresponding to different structures of the fermion sea. For every sector, the vacuum wave functions of the fermion sea must be restructured as follows

$$\psi^n_{ferm.vac.} = \Big(\prod_{k=n}^{\infty}|1_L, k\rangle\Big)\Big(\prod_{k=n-1}^{-\infty}|0_L, k\rangle\Big)\Big(\prod_{k=n-1}^{-\infty}|1_R, k\rangle\Big)\Big(\prod_{k=n}^{\infty}|0_R, k\rangle\Big), \tag{15}$$

and the vacuum wave function becomes, for every n,

$$\psi_n = \psi^n_{ferm.vac.}\psi_0\left(A_1 - \frac{2\pi}{L}n\right). \tag{16}$$

7. Show that the linear combination

$$\psi_{\theta\, vac} = \sum_n e^{in\theta}\psi_n, \tag{17}$$

is an eigenfunction of the Hamiltonian with the lowest energy independently of θ. Show also that it is invariant under large gauge transformations $A_1 \longrightarrow A_1 + 2\pi k/L$ (if we allow simultaneous renumbering of the energy levels). $\psi_{\theta\, vac}$ is therefore a gauge invariant vacuum state. The parameter θ is called the vacuum angle.

8. What is the role of the new parameter θ? Is it an observable parameter? Which term should we add to the Lagrangian of the Schwinger model, in order to imitate the presence of the vacuum angle θ?