Advanced Quantum Field Theory SS 2023

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https://www.groups.ph.tum.de/ttpmath/teaching/ss-2023/

Sheet 11: Longitudinal gauge-boson scattering and unitarity

The following problem contains a number of bonus questions, which you may attempt to solve at your own discretion.

1 Spontaneous Symmetry Breaking with different masses

Using what you know about SSB and the Higgs mechanism, construct a field theory with spontaneously broken gauge symmetry SU(2) where the masses of all 3 gauge bosons are different.

2 High energy-limit of longitudinal gauge-boson scattering and unitarity

In this problem, we will see that unitarity (via the optical theorem) implies an upper bound on the mass of the Higgs Boson. For simplicity, let's consider just the abelian $U(1)$ gauge group of hypercharge with a gauge boson B and one complex scalar Higgs field ϕ. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + (D_\mu \phi)(D^\mu \phi)^* - V(\phi)\,, \tag{1}$$

where the covariant derivative reads $D_\mu = \partial_\mu - i\, g'\, Y_\phi\, B_\mu$ with $Y_\phi > 0$ and the potential is given by

$$V(\phi) = \frac{\lambda}{4!} \left(|\phi|^2 - \frac{v^2}{2} \right)^2 . \tag{2}$$

We break the gauge symmetry spontaneously by choosing the vacuum expectation value of the Higgs field in positive real direction, $\phi = (v + h)/\sqrt{2}$, with a real scalar field h.

1. Show that the gauge boson acquires a mass $m_B^2 = (g'\, Y_\phi\, v)^2$. Derive the Feynman rules of the theory after spontaneous symmetry breaking.

Consider now the scattering process

$$B(p_1, \lambda_1) + B(p_2, \lambda_2) \quad \longrightarrow \quad B(p_3, \lambda_3) + B(p_4, \lambda_4)\,, \tag{3}$$

where p_i and λ_i denote momentum and polarisation of the corresponding gauge boson, respectively. We will write the corresponding amplitude as $\mathcal{A}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Further, as usual, we consider the process to be back-to-back and θ denotes the scattering angle, i.e. the angle between $\vec{p}_1$ and $\vec{p}_3$, in the centre-of-mass frame.

2. Show that the optical theorem implies for any configuration of polarisations the *unitarity bound*

$$\mathrm{Im}\left[\mathcal{A}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)|_{\theta=0}\right] \geq \sqrt{s(s - 4m_B^2)}\; \sigma\left(B_{\lambda_1}(p_1)\, B_{\lambda_2}(p_2) \to B_{\lambda_3}\, B_{\lambda_4}\right)\,, \tag{4}$$

where $s = (p_1 + p_2)^2 > 4m_B^2$ is the usual Mandelstam variable.

To learn something from this bound at the lowest order in perturbation theory, we could determine the imaginary part of $\mathcal{A}$ at one-loop order in forward-scattering kinematics ($\theta = 0$) and compare it to the tree-level cross section. However, it is actually not necessary to do the one-loop calculation to derive interesting implications of the unitarity bound. We will see this in the following. The first step is to view $\mathcal{A}$ as a function of $\cos\theta$, such that we can perform a *partial wave decomposition* of the amplitude,

$$\mathcal{A}(\cos\theta) = \frac{32\pi\, s}{\sqrt{s(s-4m_B^2)}} \sum_{l=0}^{\infty} (2l+1)\, A_l\, P_l(\cos\theta) \tag{5}$$

with some coefficients A_l. The overall factor in front of the sum implies a normalisation of the A_l that will turn out to be convenient later. The functions $P_l(x)$ are the Legendre polynomials you might recall from your Quantum Mechanics courses. They can be compactly written by *Rodrigues' formula*,

$$P_l(x) = \frac{1}{2^l\, l!} \frac{\mathrm{d}^l}{\mathrm{d}x^l} \left(x^2 - 1\right)^l , \tag{6}$$

fulfil $P_l(1) = 1$ and satisfy the relations

$$\sum_{l=0}^{\infty} \frac{2l+1}{2} P_l(x)\, P_l(y) = \delta(x-y) \qquad (\textit{completeness}) , \tag{7}$$

$$\int_{-1}^{1} \mathrm{d}x\, P_l(x)\, P_{l'}(x) = \frac{2\,\delta_{ll'}}{2l+1} \qquad (\textit{orthogonality}) . \tag{8}$$

3. (Bonus) The mathematically curious might be interested in proving the existence of the partial wave decomposition (5), as well as the question of how to determine the coefficients A_l in general. Given a function $f(x)$ and performing a Fourier transformation for $x = \cos\theta$,

$$f(\cos\theta) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi}\, \tilde{f}(k)\, e^{i\,k\cos\theta} , \quad \text{where} \quad \tilde{f}(k) = \int_{-\infty}^{\infty} \mathrm{d}x\, f(x)\, e^{-i\,k\,x} , \tag{9}$$

the form of eq. (5) and an explicit expression for A_l in terms of an integral follow immediately by plugging in the *partial wave expansion*

$$e^{i\,k\cos\theta} = \sum_{l=0}^{\infty} (2l+1)\, i^l\, j_l(k)\, P_l(\cos\theta) . \tag{10}$$

The functions $j_l(k)$ are the *spherical Bessel functions*. In the literature, one of the usual definitions for these functions is given by *Rayleigh's formula*,

$$j_l(k) = (-k)^l \left(\frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}k}\right)^l \frac{\sin k}{k} . \tag{11}$$

To show eq. (10), it is convenient to use instead the integral representation

$$j_l(k) = \frac{(-i)^l}{2} \int_{-1}^{1} \mathrm{d}x \, P_l(x) \, e^{i\,k\,x} \tag{12}$$

Plugging it into the right-hand side of eq. (10), the partial wave expansion simply follows from the completeness relation for the Legendre polynomials (7). However, the integral representation (12) is not quite standard and you are therefore invited here to show that it is equivalent to the textbook definition by Rayleigh's formula, eq. (11). Do this in several steps. First, write

$$\frac{\sin k}{k} = \frac{1}{2} \int_{-1}^{1} \mathrm{d}x \, e^{i\,k\,x} \tag{13}$$

and prove, for example by induction, using a suitable integration by parts, that

$$j_l(k) = \frac{(-k)^l}{2^{l+1}\, l!} \int_{-1}^{1} \mathrm{d}x \, (x^2-1)^l \, e^{i\,k\,x} \,. \tag{14}$$

Next, trade the factor $(-k)^l$ for a derivative $(i\,\mathrm{d}/\mathrm{d}x)^l$ acting on the exponential function and integrate by parts l more times to arrive at eq. (12).

After this short mathematical excursion, let's return to our scattering process and the optical theorem.

4. Go to the centre-of-mass frame and simplify the phase-space integral implicit in the right-hand side of eq.(4) such that the cross section reads

$$\sigma\left(B_{\lambda_1}(p_1)\, B_{\lambda_2}(p_2) \to B_{\lambda_3}\, B_{\lambda_4}\right) = \frac{1}{64\pi\, s} \int_{-1}^{1} \mathrm{d}(\cos\theta)\, |\mathcal{A}(\cos\theta)|^2 \tag{15}$$

5. Use this and the partial wave expansion (5) for $\mathcal{A}(\cos\theta)$ to show that the unitarity bound becomes

$$\sum_{l=0}^{\infty} (2l+1)\, \mathrm{Im}\,[A_l] \geq \sum_{l=0}^{\infty} (2l+1)\, |A_l|^2 \,. \tag{16}$$

If one considers the scattering of angular momentum eigenstates, it can be shown that the sum over l can be dropped and the unitarity bound simplifies to

$$\mathrm{Im}\,[A_l] \geq |A_l|^2 \;\; \forall l \quad \Leftrightarrow \quad \frac{1}{4} \geq (\mathrm{Re}\,[A_l])^2 + \left(\mathrm{Im}\,[A_l] - \frac{1}{2}\right)^2 \quad \forall l \quad \Rightarrow \quad \frac{1}{2} \geq |\mathrm{Re}\,[A_l]| \;\; \forall l\,. \tag{17}$$

The second bound is weaker, but it already provides a statement at lowest order in perturbation theory. So let's compute the amplitude $\mathcal{A}$ at tree-level and see what we get from it.

6. Draw the three contributing Feynman diagrams and give an expression for $\mathcal{A}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ using the Feynman rules you derived in the beginning.

As we will see soon, the coefficients A_l have a quite simple behaviour at high energies and are therefore way more easily obtained in this limit. More precisely, we consider the kinematic region

$$s = (p_1 + p_2)^2 \gg m_B^2\,, \quad |t| = |(p_1 - p_3)^2| \gg m_B^2\,, \quad |u| = |(p_1 - p_4)^2| \gg m_B^2\,. \tag{18}$$

7. (Bonus) Check that this limit exists consistent with

$$s + t + u = 4\, m_B^2\,. \tag{19}$$

Further, we haven't fixed yet the configuration of polarisations λ_i of the gauge bosons. In general, massive gauge bosons can have either one of two transverse polarizations, $\lambda = T_\pm$, or a longitudinal polarization, $\lambda = L$. In either case, the corresponding polarization vectors satisfy

$$p_\mu\, \varepsilon^\mu(p, \lambda) = 0 \quad (\textit{equation of motion})\,, \qquad \varepsilon_\mu(p, \lambda)\, \varepsilon^\mu(p, \lambda) = -1 \quad (\textit{normalisation})\,. \tag{20}$$

Additionally, in some *canonical* frame of reference, the spatial components of the transverse and longitudinal polarizations respectively fulfil the conditions

$$\vec{p} \perp \vec{\varepsilon}\,(p, T_\pm) \quad (\textit{transversality})\,, \qquad \vec{p} \parallel \vec{\varepsilon}\,(p, L) \quad (\textit{longitudinality})\,. \tag{21}$$

For the transverse polarizations, we can adopt the familiar construction from the massless case.

8. Find an explicit expression for the longitudinal polarization vector $\varepsilon^\mu(p, L)$ in the canonical frame of reference in terms of the vectors $p^\mu = (p^0, \vec{p})$ and $\tilde{p}^\mu = (p^0, -\vec{p})$.

9. Go to the centre-of-mass frame, where $\tilde{p}_1 = p_2$, $\tilde{p}_2 = p_1$, $\tilde{p}_3 = p_4$, $\tilde{p}_4 = p_3$, and argue that in the high energy limit, the scattering is strongest in the case where all gauge bosons are longitudinally polarised[1]. Studying this configuration should therefore yield the most interesting unitarity bound. Show that the corresponding amplitude behaves in this limit as

$$\mathcal{A}(L, L, L, L) = -\frac{(g'\, Y_\phi)^2}{m_B^2} \left(\frac{s^2}{s - m_h^2} + \frac{t^2}{t - m_h^2} + \frac{u^2}{u - m_h^2} \right) + \mathcal{O}\left(\left(\frac{m_B^2}{s/t/u} \right)^0 \right)\,. \tag{22}$$

Assume further that $s, |t|, |u| \gg m_h^2 \gg m_B^2$ and verify that the amplitude approaches a constant,

$$\mathcal{A}(L, L, L, L) \longrightarrow -\frac{3\, (g'\, Y_\phi)^2\, m_h^2}{m_B^2}\,. \tag{23}$$

10. Starting from $1/2 \geq |\mathrm{Re}\,[A_l]\,|$, use that there is no more dependence on the scattering angle θ in the amplitude in this limit to verify the following bound on the mass of the Higgs boson:

$$m_h \leq \sqrt{\frac{16\pi}{3}}\, v\,. \tag{24}$$

The equivalent statement in the full standard model is known as the *Lee-Quigg-Thacker bound.* Although the computation is more involved, it takes exactly the same form. This bound was the reason why it was expected that the Higgs boson or something similar would be found sooner or later at the Large Hadron Collider (LHC).

[1] Hint: Use that the transverse polarisation vectors are independent of the gauge boson mass and momentum magnitude.