Advanced Quantum Field Theory SS 2023

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https://www.groups.ph.tum.de/ttpmath/teaching/ss-2023/

Sheet 10: Quantum corrections to the linear sigma model

Part of the calculations in this exercise are a repetition of what is or will be done in class. You are nevertheless invited to repeat the calculations yourself.

1 The linear sigma model

The Lagrangian of the linear sigma model involves a set of N real scalar fields $\phi^i(x)$:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}\left[(\phi^i)^2\right]^2, \tag{1}$$

where we have replaced the usual mass term m^2 by the negative parameter $-\mu^2$ and there is an implicit sum over i in each factor of $(\phi^i)^2$. This Lagrangian is invariant under the symmetry

$$\phi^i \to R^{ij}\phi^j \tag{2}$$

for any $N \times N$ orthogonal matrix R. The group of transformations (2) is just the rotation group in N dimensions, also called the N-dimensional orthogonal group, or simply $O(N)$.

1.1 Spontaneous symmetry breaking

1. Show that any constant field ϕ_0^i that satisfies

$$(\phi_0^i)^2 = \frac{\mu^2}{\lambda} = \upsilon^2, \tag{3}$$

minimizes the potential

$$V(\phi^i) = -\frac{1}{2}\mu^2(\phi^i)^2 - \frac{\lambda}{4}\left[(\phi^i)^2\right]^2. \tag{4}$$

Eq. (3) defines an infinite set of minima with fixed modulus of the vector ϕ_0^i.

2. Suppose that the system is near one of the minima and define the fields

$$\phi^i(x) = \big(\pi^k(x),\, \upsilon + \sigma(x)\big), \quad k = 1, \ldots, N-1. \tag{5}$$

In this way you are choosing one of the minima and breaking the symmetry. This is what is usually referred to as *Spontanous Symmetry Breaking* (SSB). Rewrite the Lagrangian (1) in terms of the π and σ fields and show that

$$\begin{aligned}\mathcal{L} =& \frac{1}{2}(\partial_\mu \pi^k)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 \\ &- \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu(\pi^k)^2\sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}(\pi^k)^2\sigma^2 - \frac{\lambda}{4}\left[(\pi^k)^2\right]^2.\end{aligned} \tag{6}$$

Explain why this Lagrangian describes a massive σ field with mass $\sqrt{2}\mu$ and also a set of $N-1$ massless π fields.

3. Derive the Feynman rules from the Lagrangian after SSB and prove that they can be written as in Figure 1.

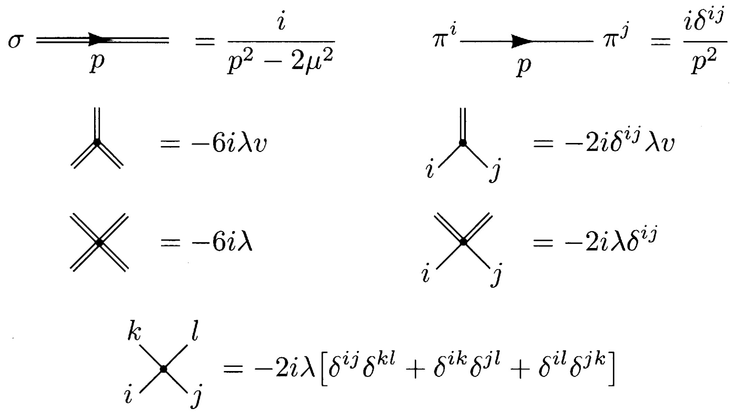


Figure 1: Feynman rules for the linear sigma model.

2 One-loop corrections and renormalization

In this part of the exercise we want to study the renormalization of this theory to one-loop order. As always, for an amplitude with N_e external legs, the so-called *superficial degree of divergence* in 4 dimensions is defined as

$$D = 4 - N_e. \tag{7}$$

2.1 Counterterms

1. Enumerate all the superficially divergent amplitudes in the linear sigma model.

2. What are the bare parameters available to absorb the infinities? Rewrite the original Lagrangian before SSB in terms of physical parameters and counterterms in terms of π and σ fields. Show that the counter-term Lagrangian after SSB takes the following form,

$$\begin{aligned}\frac{\delta_Z}{2}(\partial_\mu\pi^k)^2 - \frac{1}{2}(\delta_\mu + \delta_\lambda v^2)(\pi^k)^2 + \frac{\delta_Z}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}(\delta_\mu + 3\delta_\lambda v^2)(\sigma)^2 \\ - (\delta_\mu v + \delta_\lambda v^3)\sigma - \delta_\lambda v\sigma(\pi^k)^2 - \delta_\lambda v\sigma^3 \\ - \frac{\delta_\lambda}{4}\big[(\pi^k)^2\big]^2 - \frac{\delta_\lambda}{2}\sigma^2(\pi^k)^2 - \frac{\delta_\lambda}{4}\sigma^4,\end{aligned} \tag{8}$$

   where δ_Z, δ_μ, δ_λ are the shifts in the field strength, mass and coupling constant respectively.

The Feynman rules for the counterterms (8) are given in Figure 2. Notice that while there is a large number of vertices and counter terms, the symmetry of the original Lagrangian implies that they all depend only on three independent parameters δ_λ, δ_μ and δ_Z.

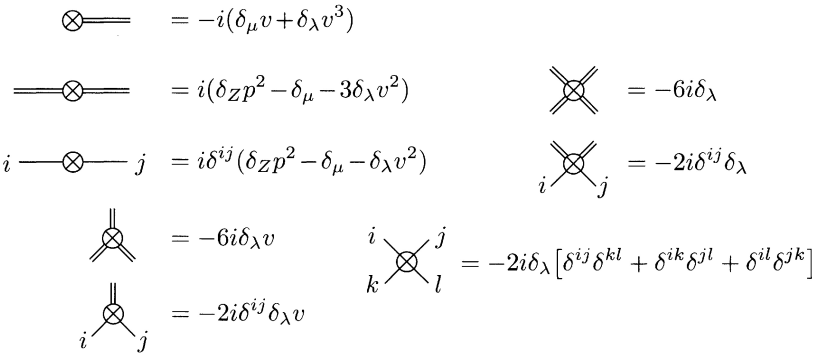
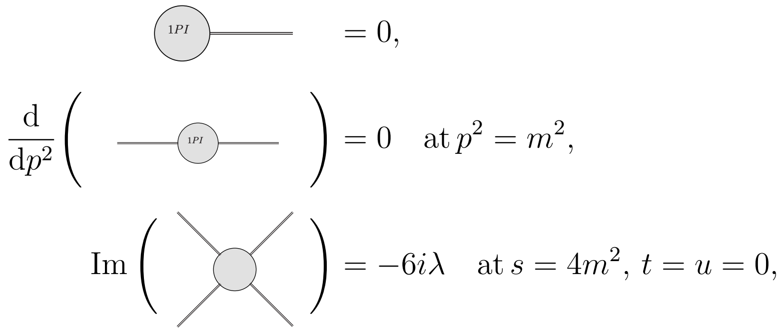


Figure 2: Feynman rules for the counterterm vertices.

2.2 Renormalization conditions and fixing counterterms

We want to show now that by fixing only these three counterterms, all divergent Green functions are regularized. As renormalization conditions we can choose the following:



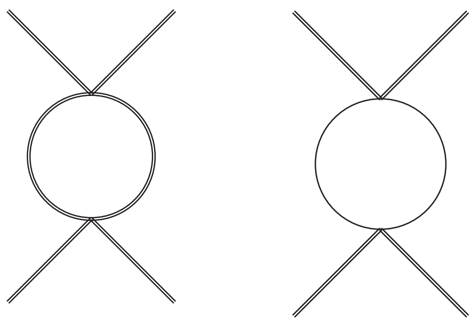
$$\text{(1PI tadpole)} = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}p^2}\Big(\text{(1PI two-point)}\Big) = 0 \quad \text{at}\, p^2 = m^2, \tag{9}$$

$$\mathrm{Im}\Big(\text{(four-point)}\Big) = -6i\lambda \quad \text{at}\, s = 4m^2,\ t = u = 0,$$

1. Start by determining the counterterm δ_λ from the 4σ amplitude:

   (a) Draw the contributing one-loop diagrams and argue that the only divergent diagrams are of the type



(10)

   (b) Compute the above diagrams and show that the counterterm δ_λ is given by

$$\delta_\lambda \sim \lambda^2(N+8)\frac{\Gamma(2-d/2)}{(4\pi)^2}, \tag{11}$$

where we have omitted finite terms.

(c) Show that δ_λ renders the remaining superficially divergent four-point amplitudes ($\sigma\sigma\pi\pi$ and 4π) finite.

2. Show that δ_λ renders finite the three-point amplitudes, $\sigma\sigma\sigma$ and $\sigma\pi\pi$.

3. Apply the first renormalization condition from (9) and show that

$$(\delta_\mu + \upsilon^2 \delta_\lambda) = -\lambda \frac{\Gamma(1-d/2)}{(4\pi)^{d/2}} \left(\frac{3}{(2\mu^2)^{1-d/2}} + \frac{N-1}{(\zeta^2)^{1-d/2}} \right). \tag{12}$$

When applying (9) at one-loop order you will have to deal a tadpole that involves the massless π field running inside the loop. This diagram involves a divergent integral over a massless propagators. To make sense out of it, you can introduce a small mass ζ for π field as an infrared regulator.

4. (Bonus) Compute the 2σ amplitude and show that $\delta_Z = 0$.