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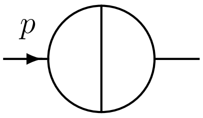
https://www.groups.ph.tum.de/ttpmath/teaching/ss-2023/

Sheet 01: Loop integrals and Cutkosky rules

Problem 1 - A two-loop massless bubble

The goal of this exercise is to get some familiarity with dimensional regularisation and with the idea of Integration-By-Parts identities (IBP's). We will use them to calculate analytically the following two-loop (!) integral in dimensional regularisation ($d = 4 - 2\epsilon$)

$$\mathcal{I}(p^2) =$$



$$= \int \frac{d^d q_1}{(2\pi)^d} \int \frac{d^d q_2}{(2\pi)^d} \, \frac{1}{q_1^2\,(q_1-p)^2\,(q_1-q_2)^2\,q_2^2\,(q_2-p)^2} \,. \tag{1}$$

This integral is finite for $\epsilon = 0$ and is given by the very simple expression

$$\mathcal{I}(p^2) = -\frac{6\,\zeta_3}{(4\,\pi)^4\,p^2} + \mathcal{O}(\epsilon)\,.$$

However, deriving this result by means of direct integration over Feynman parameters is difficult. IBP's provide instead a much more elegant way.

1. Start off by performing a Wick rotation in order to go to the euclidean region

$$q_1^0 = -i\,k^0\,, \qquad q_2^0 = -i\,l^0\,, \qquad p_0 = -i\,p_E^0 \tag{2}$$

   such that the integral becomes

$$\mathcal{I}(p^2) = \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} \, \frac{1}{k^2\,(k-p_E)^2\,(k-l)^2\,l^2\,(l-p_E)^2} = \mathcal{I}_E(p_E^2)\,, \tag{3}$$

   where the vectors $k,\ l$ are Euclidean, i.e. $k^2 = k_0^2 + \vec{k}^2\,,\ l^2 = l_0^2 + \vec{l}^2$ and $p_E^2 = -p^2$.

2. Let us focus now on the euclidean integral $\mathcal{I}_E(p_E^2)$. Argue why, in dimensional regularisation, we can write

$$\int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} \, \left[\frac{\partial}{\partial k_\mu} v_\mu \frac{1}{k^2\,(k-p_E)^2\,(k-l)^2\,l^2\,(l-p_E)^2}\right] = 0\,, \tag{4}$$

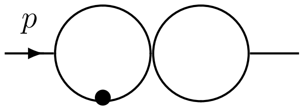
   for every vector $v_\mu = k_\mu, l_\mu, p_{E,\mu}$. This type of relations are referred to as Integration-By-Parts Identities, or IBPs.

3. Specialise the IBP above choosing $v_\mu = k_\mu - l_\mu$ and use it to prove that the integral $\mathcal{I}_E(p_E^2)$ can be *reduced* as

$$\mathcal{I}_E(p_E^2) = \frac{2}{d-4}\left(\mathcal{I}_1(p_E^2) - \mathcal{I}_2(p_E^2)\right)\,, \tag{5}$$

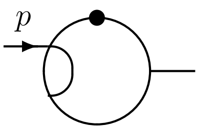
where:

$$\mathcal{I}_1(p_E^2) =$$



$$= \int \frac{d^dk}{(2\pi)^d}\int \frac{d^dl}{(2\pi)^d}\,\frac{1}{k^4\,(k-p_E)^2\,l^2\,(l-p_E)^2}\,, \tag{6}$$

$$\mathcal{I}_2(p_E^2) =$$



$$= \int \frac{d^dk}{(2\pi)^d}\int \frac{d^dl}{(2\pi)^d}\,\frac{1}{k^4\,(k-p_E)^2\,(k-l)^2\,(l-p_E)^2}\,. \tag{7}$$

4. You need now to compute the integrals $\mathcal{I}_1(p_E^2)$ and $\mathcal{I}_2(p_E^2)$. Start off by defining the Euclidean one-loop bubble with arbitrary powers of propagators

$$\mathcal{B}(q_E^2;a,b) = \int \frac{d^dk}{(2\pi)^d}\,\frac{1}{(k^2)^a\,\left((k-q_E)^2\right)^b}\,. \tag{8}$$

Using Feynman parameters prove that

$$\mathcal{B}(q_E^2;a,b) = \frac{(4\pi)^\epsilon}{16\pi^2}\,\frac{\Gamma(2-\epsilon-a)\,\Gamma(2-\epsilon-b)\,\Gamma(a+b-2+\epsilon)}{\Gamma(a)\,\Gamma(b)\,\Gamma(4-2\epsilon-a-b)}\,\left(q_E^2\right)^{2-\epsilon-a-b}\,, \tag{9}$$

where as usual $d = 4-2\epsilon$.

5. Using only eq. (9) and defining

$$S_\epsilon = \frac{(4\pi)^\epsilon\,\Gamma(1+\epsilon)\,\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\,,$$

prove that[1]:

$$\mathcal{I}_1(p_E^2) = \left(\frac{S_\epsilon}{16\pi^2}\right)^2\left(-\frac{1}{\epsilon^2(1-2\epsilon)}\right)\left(p_E^2\right)^{-1-2\epsilon}\,, \tag{10}$$

$$\mathcal{I}_2(p_E^2) = \left(\frac{S_\epsilon}{16\pi^2}\right)^2\left(-\frac{1}{\epsilon^2(1-2\epsilon)}\right)\frac{\Gamma(1-2\epsilon)^2\,\Gamma(1+2\epsilon)}{\Gamma(1-\epsilon)\,\Gamma(1+\epsilon)^2\,\Gamma(1-3\epsilon)}\left(p_E^2\right)^{-1-2\epsilon}\,. \tag{11}$$

[1] Make use, where necessary, of the functional identity $\Gamma(1+x) = x\,\Gamma(x)$ in order to extract explicitly all poles in $1/\epsilon$.

6. Using the series expansion

$$\Gamma(1+n\epsilon)\, e^{n\,\gamma\,\epsilon} = 1 + \frac{\pi^2}{12} n^2\epsilon^2 - \frac{\zeta_3}{3} n^3 \epsilon^3 + \mathcal{O}(\epsilon^4)\,,$$

where γ is the Euler-Mascheroni constant, expand all Γ functions around $\epsilon = 0$ and prove that

$$\frac{\Gamma(1-2\epsilon)^2\,\Gamma(1+2\epsilon)}{\Gamma(1-\epsilon)\,\Gamma(1+\epsilon)^2\,\Gamma(1-3\epsilon)} = 1 - 6\,\zeta_3\,\epsilon^3 + \mathcal{O}(\epsilon^4)\,. \tag{12}$$

7. Finally putting everything together show that

$$\mathcal{I}_E(p_E^2) = \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \,(\,6\,\zeta_3 + \mathcal{O}(\epsilon))\,\left(p_E^2\right)^{-1-2\epsilon} \tag{13}$$

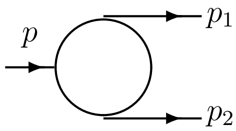
such that in the minkowskian, physical, region we have:

$$\mathcal{I}(p^2) = \left(\frac{S_\epsilon}{16\pi^2}\right)^2 \,(\,6\,\zeta_3 + \mathcal{O}(\epsilon))\,\left(-p^2 - i\,\delta\right)^{-1-2\epsilon} = -\frac{6\,\zeta_3}{(4\,\pi)^4\,p^2} + \mathcal{O}(\epsilon)\,, \tag{14}$$

where $0 < \delta \ll 1$ comes from Feynman's prescription.

Problem 2 - The imaginary part of a 1-loop triangle

Consider the following 1-loop triangle with massless internal propagators and massive external lines



$$\mathcal{T}(p^2, m_1, m_2) = [\text{diagram}] = \int \frac{d^4\,k}{(2\pi)^4}\,\frac{1}{k^2\,(k-p_1)^2\,(k-p_1-p_2)^2}\,, \qquad p_1^2 = m_1^2\,, \quad p_2^2 = m_2^2\,. \tag{15}$$

In the decay kinematics one has $p \to p_1 + p_2$ with $p^2 \geq (m_1+m_2)^2$. The integral $\mathcal{T}(p^2, m_1, m_2)$ is finite so that no regularization is required.

1. Use Feynman parameters to show that $\mathcal{T}(p^2, m_1, m_2)$ for this choice of the kinematics, cannot develop any imaginary part. *Note: You do not need to compute the integral explicitly, but only to show that it must be real!*

2. We want to compute now this imaginary part explicitly using Cutkosky rules. Start off by computing the discontinuity in p^2, i.e. the one obtained by cutting the two propagators connecting to the vertex with momentum p. Show that this discontinuity reads

$$\mathrm{Disc}(p^2, m_1, m_2) = \frac{1}{8\,\pi\,\sqrt{(p^2-\mu_{12}^2)\,(p^2-\bar{\mu}_{12}^2)}}\ln\left(\frac{p^2-m_1^2-m_2^2-\sqrt{(p^2-\mu_{12}^2)\,(p^2-\bar{\mu}_{12}^2)}}{p^2-m_1^2-m_2^2+\sqrt{(p^2-\mu_{12}^2)\,(p^2-\bar{\mu}_{12}^2)}}\right), \tag{16}$$

with $\mu_{12} = (m_1 + m_2)$ and $\bar{\mu}_{12} = (m_1 - m_2)$. It is convenient to work in the reference frame where p^μ is at rest, namely $p^\mu = (W, \vec{0}\,)$, where $W = \sqrt{p^2}$ is the total energy of the system.

3. Compute now the discontinuity in $p_1^2 = m_1^2$ by cutting the two propagators connecting to the vertex with momentum p_1. In this case it is convenient to work in the reference frame where p_1 is at rest, namely $p_1^\mu = (m_1, \vec{0}\,)$. Show that in this case one gets

$$\mathrm{Disc}(m_1, p^2, m_2) = \frac{1}{8\,\pi\,\sqrt{(p^2-\mu_{12}^2)\,(p^2-\bar{\mu}_{12}^2)}}\ln\left(\frac{p^2+m_2^2-m_1^2+\sqrt{(p^2-\mu_{12}^2)\,(p^2-\bar{\mu}_{12}^2)}}{p^2+m_2^2-m_1^2-\sqrt{(p^2-\mu_{12}^2)\,(p^2-\bar{\mu}_{12}^2)}}\right), \tag{17}$$

4. Finally the third cut does not need to be computed and can be obtained just permuting m_1 and m_2 in (17)

$$\mathrm{Disc}(m_2, p^2, m_1) = \frac{1}{8\,\pi\,\sqrt{(p^2-\mu_{12}^2)\,(p^2-\bar{\mu}_{12}^2)}}\ln\left(\frac{p^2+m_1^2-m_2^2+\sqrt{(p^2-\mu_{12}^2)\,(p^2-\bar{\mu}_{12}^2)}}{p^2+m_1^2-m_2^2-\sqrt{(p^2-\mu_{12}^2)\,(p^2-\bar{\mu}_{12}^2)}}\right). \tag{18}$$

5. Show then that the total imaginary part must is, *as expected* from 1., zero

$$\frac{1}{\pi}\,\mathrm{Im}\left(\mathcal{T}(p^2, m_1, m_2)\right) = \mathrm{Disc}(p^2, m_1, m_2) + \mathrm{Disc}(m_1, p^2, m_2) + \mathrm{Disc}(m_2, p^2, m_1) = 0\,. \tag{19}$$