The Drell-Yan process @ NLO QCD

We are interested in the hadroproduction of a pair of leptons, a process commonly referred to as the Drell-Yan process. There is a neutral-current (NCDY) and charged-current (CCDY)

$$\text{NCDY:}\ h_1 h_2 \to \gamma^*/Z \to \ell^- \ell^+ \qquad \text{CCDY:}\ \begin{array}{l} h_1 h_2 \to W^+ \to \ell^+ \nu_\ell \\ h_1 h_2 \to W^- \to \ell^- \bar{\nu}_\ell \end{array} \tag{1}$$

where $h_{1,2}$ are hadrons. For definiteness we consider protons, and for the time being on NCDY (most of the following discussion applies, unchanged to $W^\pm$ production). This reaction is the LHC's flagship precision channel and provide invaluable information on several aspect of the SM and its phenomenological investigations at hadron colliders:

- allows for measurements and constraints on PDFs
- allows for precise measurements of Electroweak parameters such as: M_W, $\sin\theta_W$, but also M_Z
- it's a standard candle process, allowing for monitoring of the collider luminosity and for detector calibration.

When comparing theoretical predictions to experimental measurements the basic quantity is the hadronic cross-section. In the experiments, one is able to fully reconstruct the momenta/kinematics of the identified leptons, and thank to this to construct distributions for observable of interest. Some examples are:

- invariant mass of the lepton pair
- transverse momentum of the individual leptons or of the pair
- rapidity of the individual leptons or of the pair.

The master formula for the hadronic cross section can be written in factorised form:

$$\sigma_n = \sum_{a,b} \int_0^1 dx_a \, dx_b \underbrace{f_{a/h_1}(x_a) \, f_{b/h_2}(x_b)}_{\text{parton distribution functions (PDFs)}} \underbrace{\hat{\sigma}_{ab}(\Phi_n)}_{\text{partonic cross-section}} \quad \Leftarrow h_1, h_2 \to n \text{ particles} \qquad (2)$$

where a, b label the partons species $(g, q, \bar{q})$ and $\hat{\sigma}_{ab}$ is the partonic cross-section for a given process initiated by partons a, b. This can be schematically written as:

$$\hat{\sigma}_{ab\to n} = \frac{1}{2\hat{s}} \int d\Phi_n \, |\mathcal{M}_{ab\to n}|^2 (\Phi_n) \qquad (3)$$

$d\Phi_n \to$ Lorentz-invariant phase space for an n-body configuration

Since we will work in dimensional-regularisation, this reads

$$d\Phi_n = \left[\prod_{i=1}^{n} \frac{d^D p_i}{(2\pi)^D} (2\pi) \, \delta(p_i^2 - m_i^2) \, \theta(p_i^0) \right] (2\pi)^D \, \delta^{(D)}\Big(p_a + p_b - \sum_{i=1}^{n} p_i\Big) . \qquad (4)$$

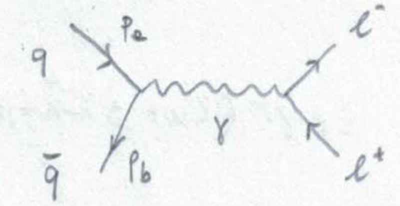
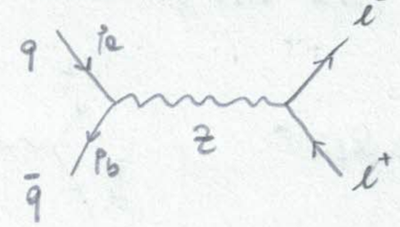
In Eq. (3) $|\mathcal{M}_{ab\to n}|^2$ is the squared matrix element for the scattering process $p_a + p_b \to q_1 + \ldots + p_n$. The overall $(2\hat{s})^{-1}$ term is the flux factor. Eventually, we'll be interested in providing predictions for specific distributions of observable $\mathcal{O}_i \in \{q_T, y_\ell, p_{T,\ell} \ldots\}$. Therefore we denote the differential cross-section wrt some quantities $\mathcal{O}_1, \ldots, \mathcal{O}_m$ as:

$$\frac{d\sigma}{d\mathcal{O}_1 \ldots d\mathcal{O}_m} \xrightarrow{\text{example}} \frac{d\sigma}{dq_T \, dy_\ell} \Rightarrow$$ double-differential cross section wrt the transverse momentum and rapidity of the lepton pair.

Ultimately, we will be interested in fully differential cross-sections, namely that we are able to provide a prediction for any (well-defined) observable built out of the kinematics of the identified final-state objects.

A) Cross-section at LO

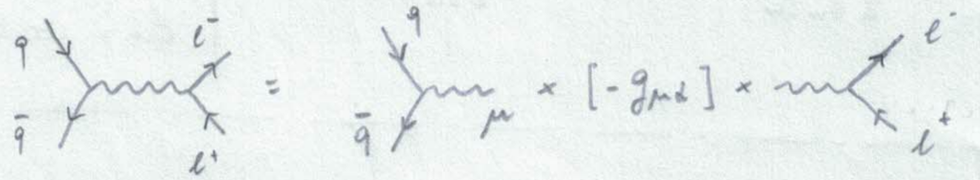
At leading-order (LO) there's only a quark-antiquark annihilation channel, and the contributing diagrams are:

$\mathcal{M} =$  $+$  (5)

The scattering amplitude $\mathcal{M}$ can be conveniently written as a production times a decay stage, i.e. (we work with massless leptons)

$$\mathcal{M} = \sum_{V=\gamma,Z} H_V^{\mu} \left[\frac{-i\, g_{\mu\alpha}}{Q^2 - m_V^2 + i\Gamma_V m_V} \right] L_V^{\alpha} \qquad (6)$$

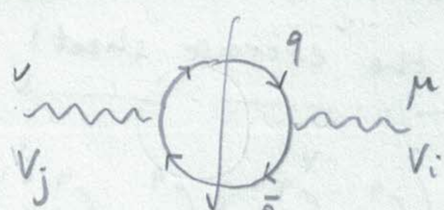
with $m_\gamma = \Gamma_\gamma = 0$ and $Q = p_\ell + p_{\bar\ell}$ is the momentum of the intermediate vector boson. Pictorially we are doing

 (7)

At squared-matrix element level (summed over the polarisations of external particles)

$$\sum_{pol} |\overline{\mathcal{M}}_{ab\to\ell\bar\ell}|^2 = \sum_{V_i,V_j=\gamma,Z} \mathcal{H}^{\mu\nu}_{V_iV_j}\, \Delta_{V_iV_j}\, \mathcal{L}_{\mu\nu,V_iV_j} \qquad (8)$$

where we have introduced

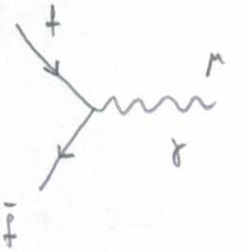
$$\mathcal{H}^{\mu\nu}_{V_iV_j} = \sum_{pol} H^{\mu}_{V_i} H^{\nu *}_{V_j} \to$$  $\Rightarrow$ summed over quark polarisations

$$\mathcal{L}^{\mu\nu}_{V_iV_j} = \sum_{pol} L^{\mu}_{V_i} L^{\nu *}_{V_j} \Rightarrow$$ summed over lepton polarisations (9)

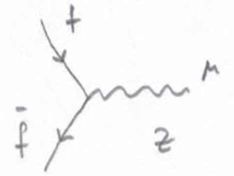
$$\Delta_{V_iV_j} = \frac{1}{[Q^2 - m_{V_i}^2 + i\Gamma_{V_i} m_{V_i}][Q^2 - m_{V_j}^2 - i\Gamma_{V_j} m_{V_j}]}$$

Therefore, also at squared matrix-element square we have a factorised pattern.

The relevant Feynman rules we need for computing the amplitudes (and also for later when we consider QCD corrections) are as follows

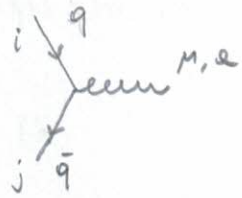


$= -ieQ_f \gamma^\mu$

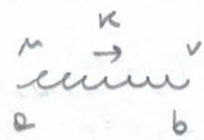


$= ie\gamma^\mu \left(C^f_{V,Z} + C^f_{A,Z}\gamma^5 \right)$

(10)



$= -i g_s \gamma^\mu T^a_{ij}$



$= -i \dfrac{g^{\mu\nu}}{k^2} \delta^{ab}$.

with C_V, C_A the vector and axial couplings given by:

$$C^f_{V,Z} = \frac{I^f_3 - 2Q_f s_w^2}{2c_w s_w}, \quad C^f_{A,Z} = -\frac{I^f_3}{2 s_w c_w} \tag{11}$$

$I_3 \to$ 3rd component of isospin

$s_w = \sin\theta_w$

$c_w = \cos\theta_w$

Exercise:

compute the interference terms $\mathcal{H}^{\mu\nu}$ and $L^{\alpha\beta}$, working in conventional dimensional regularisation (CDR), namely carrying out the Lorentz and Dirac algebra in D-dimensions. Also for the hadronic (production) part, sum and average over colour and spins. (more details on the exercise sheet)

$$\mathcal{H}^{\mu\nu}_{V_iV_j} = \frac{4\pi\alpha_w}{N_c}\Big[\left(C^q_{V,V_i}C^q_{V,V_j} + C^q_{A,V_i}C^q_{A,V_j}\right)\left(p_a^\mu p_b^\nu + p_a^\nu p_b^\mu - g^{\mu\nu} p_a\cdot p_b\right) + \left(C^q_{V,V_i}C^q_{A,V_j} + C^q_{A,V_i}C^q_{V,V_j}\right) i\,\varepsilon^{\mu\nu\rho\sigma} p_{a,\rho}\, p_{b,\sigma}\Big] \tag{12}$$

where $C^f_{A,\gamma} = 0$ and $C^f_{V,\gamma} = Q_f$.

The leptonic tensor has a similar structure, the only difference lies in the couplings ($q \leftrightarrow \ell$), the momenta ($p_{a,b} \leftrightarrow p_\ell, p_{\bar\ell}$) and an overall colour factor.

Let us now turn to the calculation of the partonic cross-section, for now fully differential:

$$d\sigma = \frac{1}{2\hat{s}} \frac{d^Dp_{\ell^-}}{(2\pi)^D} \frac{d^Dp_{\ell^+}}{(2\pi)^D} (2\pi)^2 \delta(p_{\ell^+}^2)\, \delta(p_{\ell^-}^2)\, (2\pi)^D \delta^D(p_a + p_b - p_{\ell^-} - p_{\ell^+})\, |\mathcal{M}_{ab\to\ell\ell'}|^2 \tag{13}$$

massless leptons

we can introduce unity in the measure via

$$1 = \int_0^\infty \frac{dM^2}{2\pi} (2\pi)\, \delta(Q^2 - M^2) \frac{d^DQ}{(2\pi)^D} (2\pi)^D \delta^{(D)}(Q - p_{\ell^-} - p_{\ell^+}) \tag{14}$$

which allows to factorize the phase space as well, leading to

$$\begin{aligned} d\sigma = \sum_{V_iV_j} \frac{1}{2\hat{s}} \int_0^\infty \frac{dM^2}{2\pi} \int \frac{d^DQ}{(2\pi)^D} (2\pi)\, \delta(Q^2 - M^2)\, (2\pi)^D \delta^D(p_a + p_b - Q)\, H^{\mu\nu}_{V_iV_j}\, \Delta_{V_iV_j} \\ \times \frac{d^Dp_{\ell^-}}{(2\pi)^D} \frac{d^Dp_{\ell^+}}{(2\pi)^D} (2\pi)^2 \delta(p_{\ell^-}^2)\, \delta(p_{\ell^+}^2)\, (2\pi)^D \delta^{(D)}(Q - p_{\ell^-} - p_{\ell^+})\, L_{\mu\nu} \end{aligned} \tag{15}$$

Hence it is a factorised production and subsequent decay of a vector boson of momentum Q. For the moment, let us integrate over the leptons phase space (we are interested in observables corresponding to the intermediate boson). This integration only affects the second line of Eq. (15), namely:

$$\int \frac{d^Dp_{\ell^-}}{(2\pi)^D} \frac{d^Dp_{\ell^+}}{(2\pi)^D} (2\pi)^2 \delta(p_{\ell^-}^2)\, \delta(p_{\ell^-}^2)\, (2\pi)^D \delta^{(D)}(Q - p_{\ell^-} - p_{\ell^+})\, L_{\mu\nu} =: T_{\mu\nu} \tag{16}$$

Since we are integrating over the lepton momenta, $T_{\mu\nu}$ is a rank-2 tensor that can be built out of $g_{\mu\nu}$, Q_μ, Q_ν and $\varepsilon^{\mu\nu\rho\sigma}$ only. However, the latter is ruled out since it can only be contracted with Q_ρ, Q_σ. We are thus left with:

$$T_{\mu\nu} = A\, g_{\mu\nu} + B \frac{Q_\mu Q_\nu}{Q^2} \tag{17}$$

We note that the leptonic tensor $L_{\mu\nu}$ is transverse to the vector boson current, i.e.

$$Q_\mu L^{\mu\nu} = Q_\nu L^{\mu\nu} = 0 \tag{18}$$

which gives us $\quad 0 = Q_\mu T^{\mu\nu} = Q_\mu \left(A g^{\mu\nu} + B \frac{Q^\mu Q^\nu}{Q^2} \right) \Rightarrow A = -B.$

We conclude that the fully inclusive leptonic component is given by:

$$T^{\mu\nu} = F_{ll} \left(-g^{\mu\nu} + \frac{Q^\mu Q^\nu}{Q^2} \right) \tag{19}$$

<u>Exercise</u>: compute the fully inclusive leptonic phase-space + tensor and extract the form factor F_{ll}.

Imagine now we are interested in studying properties of the Z boson, namely we look into an invariant mass range which removes the contribution from the γ-pole, i.e. assume $m_{ll} \gtrsim$ few GeVs.

The dominant contribution to the cross-section is expected from a resonant Z boson $\Rightarrow \quad \Delta_{ij} \sim \frac{1}{(Q^2 - m_Z^2)^2 + \Gamma_Z^2 m_Z^2}$. Further, we observe that $\Gamma_Z \ll m_Z$ $\quad (\Gamma_Z \sim 2.49\ \text{GeV},\ m_Z \sim 91.1876\ \text{GeV})$.

If we now consider only the integration over the invariant mass M^2 we have (considering the dominant resonance region)

$$\int_{m^2_{min}}^{\infty} \frac{dM^2}{2\pi} \frac{1}{\left[(Q^2 - m_Z^2)^2 + \Gamma_Z^2 m_Z^2 \right]} \xrightarrow{\Gamma_Z \ll m_Z} \int_{m^2_{min}}^{\infty} \frac{dM^2}{2\pi} \frac{\pi}{\Gamma_Z m_Z} \delta(M^2 - m_Z^2) + \mathcal{O}\left(\frac{\Gamma_Z}{m_Z}\right) \tag{20}$$

$\Uparrow$

narrow width approximation (NWA).

Therefore, if we integrate over the invariant mass spectrum and work in the NWA, we are effectively considering the production of on-shell Z boson which subsequently decays into a lepton pair.

If we look indeed at Eq. (19) the tensor structure $T^{\mu\nu}$ becomes:

$$T^{\mu\nu} = F_{\ell\ell}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{m_Z^2}\right) = F_{\ell\ell} \sum_{\text{pol}} \varepsilon_Z^\mu(q)\, \varepsilon_Z^{\nu *}(q) \tag{21}$$

Sum over the polarisations of a physical (on-shell) Z-boson

$\Rightarrow$ combining the result from the dM^2 integral with the one over the leptons d.o.f we obtain:

$$d\hat{\sigma}_{NWA} = \frac{1}{2\hat{s}} \int \frac{d^D q}{(2\pi)^D} (2\pi)\, \delta(q^2 - M_Z^2)\, (2\pi)^D \delta^D(p_a + p_b - q)\, |M_{ab\to Z}|^2 \tag{22}$$

$$\times \frac{1}{\Gamma_Z} \times \underbrace{\frac{1}{2m_Z} \int \frac{d^D p_{\ell^-}}{(2\pi)^D} \frac{d^D p_{\ell^+}}{(2\pi)^D} (2\pi)^2 \delta(p_{\ell^-}^2)\, \delta(p_{\ell^+}^2)\, F_{\ell\ell}\, (2\pi)^D \delta^{(D)}(q - p_{\ell^-} - p_{\ell^+})}_{\Gamma_{Z\to \bar{\ell}\ell^+} \text{ (Partial decay width)}}$$

$$\Rightarrow d\hat{\sigma}_{NWA} = d\hat{\sigma}_{ab\to Z} \times Br(Z \to \ell^-\ell^+). \tag{23}$$

In the following discussion we can discard the branching ratio (we are interested in the "QCD component"). Thus, from now on we will consider the cross-section for the production of an on-shell Z-boson.

In the lab frame, i.e. centre of mass frame of the scattering protons, we can parametrize the kinematics as follows:

$$\begin{cases} p_a = x_a \frac{\sqrt{S}}{2}(1, \vec{0}^{\,d-2}, 1), \quad p_b = x_b \frac{\sqrt{S}}{2}(1, \vec{0}^{\,d-2}, -1) \Rightarrow p_a \cdot p_b = x_a x_b \frac{S}{2} \\ q = (E_T \cosh y, \vec{p}_T^{\,d-2}, E_T \sinh y) \Rightarrow y: \text{Z-boson rapidity in the lab frame.} \end{cases} \tag{24}$$

The momentum-conserving δ-functions in the phase space give:

$$q = p_a + p_b \Rightarrow m_Z^2 = (p_a + p_b)^2 \Rightarrow x_a x_b = \frac{m_Z^2}{S} := \tau \tag{25}$$

As for the squared matrix element we have $(D = 4-2\varepsilon)$

$$|\mathcal{M}_{ab\to Z}|^2 = \frac{4\pi\alpha_W}{N_c}(c_V^2 + c_A^2)\, m_Z^2 (1-\varepsilon) = \sigma_{LO} \cdot \frac{S m_Z^2}{\pi} \tag{26}$$

where for future convenience we have introduce:

$$\sigma_{LO} = \frac{4\pi^2\alpha_W}{N_c S}(c_A^2 + c_V^2) \times (1-\varepsilon) \tag{27}$$

As for the partonic fractions x_a, x_b we observe that $\hat{S} \geq m_Z^2 \Rightarrow x_a x_b S \geq m_Z^2 \Rightarrow x_a x_b \geq \tau$, but also $0 \leq x_{a,b} \leq 1$ therefore we have the boundaries $\tau \leq x_a, x_b \leq 1$. For the hadronic cross-section we thus get:

$$\sigma_{pp\to Z} = \int_\tau^1 dx_a\, dx_b\, f_a(x_a) f_b(x_b) \int \frac{d^D Q}{(2\pi)^D} (2\pi)\delta(Q^2 - m_Z^2)(2\pi)^D \delta^D(p_a + p_b - Q) \times \frac{1}{2\hat{S}} \sigma_{LO} \cdot \frac{S m_Z^2}{\pi}$$

$$= \sigma_{LO} \int_\tau^1 dx_a\, dx_b\, f_a(x_a) f(x_b)\, \delta(x_a x_b - \tau) \tag{28}$$

combining all combination of $(q\bar{q})$ pairs from h_1, h_2 we have

$$\sigma_{pp\to Z} = \sum_{ab} \sigma_{LO}^{ab} \int_\tau^1 dx \left[f_{q/h_1}(x) f_{\bar{q}/h_2}\left(\frac{\tau}{x}\right) + f_{\bar{q}/h_1}(x) f_{q/h_2}\left(\frac{\tau}{x}\right) \right] \tag{29}$$

Imagine we want to be differential wrt y and p_T. We can write

$$\sigma_{pp\to Z} = \int dy\, dp_T^2 \underbrace{\frac{d^2\sigma_{pp\to Z}}{dy\, dp_T^2}} \tag{30}$$

double differential cross-section

The kinematic constraint $p_a + p_b = Q$ gives us:

$p_T^2 = 0$ and $y = \frac{1}{2}\log\left(\frac{x_a}{x_b}\right)$ which combined with $x_a x_b = \tau$ give us the partonic fractions in terms of the mass and the rapidity of the Z-boson i.e.

$$x_a = \sqrt{\tau}\, e^{-y} \qquad\qquad x_b = \sqrt{\tau}\, e^{-y} \tag{31}$$

we can then introduce a unity inside Eq. (28) as

$$1 = \int_0^{\infty} dp_T^2\, \delta(p_T^2) \times \int_{y_{min}}^{y_{max}} dy\, \delta\left(y - \frac{1}{2}\log\left(\frac{x_a}{x_b}\right)\right) \tag{32}$$

where y_{min} and y_{max} $y_{min} = \frac{1}{2}\log\tau$, $y_{max} = -\frac{1}{2}\log\tau$.

Eq. (28) then becomes:

$$\sigma_{LO}\int_{\tau}^{1} dx_a\, dx_b\, f_a(x_a)\, f_b(x_b) \int_0^{\infty} dp_T^2\, \delta(p_T^2) \int_{y_{min}}^{y_{max}} dy\, \delta\left(y - \frac{1}{2}\log\left(\frac{x_a}{x_b}\right)\right) \times \delta(x_a x_b - \tau) \tag{33}$$

Now we can integrate over x_a and x_b and get:

$$\Rightarrow \quad \sigma_{LO} \times \int_0^{\infty} dp_T^2 \int_{y_{min}}^{y_{max}} dy \times f_a(x_a)\, f_b(x_b)\, \delta(p_T^2) \Bigg|_{\substack{x_a = \sqrt{\tau}\, e^{y} \\ x_b = \sqrt{\tau}\, e^{-y}}} \tag{34}$$

which has the desired form of Eq. (30). The double-diff. cross section is finally

$$\frac{d\sigma_{pp\to Z}}{dy\, dp_T^2} = \sum_{a,b} \sigma_{LO}^{ab}\, \delta(p_T^2) \left[f_{q/h_1}(x_a)\, f_{\bar{q}/h_2}(x_b) + f_{\bar{q}/h_1}(x_a)\, f_{q/h_2}(x_b) \right] \Bigg|_{\substack{x_a = \sqrt{\tau}\, e^{y} \\ x_b = \sqrt{\tau}\, e^{-y}}}.$$

We then see that the rapidity distribution of the Z-boson gives valuable information about the quark PDFs.

*) Cross-section at NLO QCD: Virtual corrections

Here we limit ourselves to report the result of the 1-loop QCD form factor. Its explicit computation is left as an exercise. For the actual cross-section we'll need the tree-1loop interference

$$\mathcal{M}_{q\bar{q}\to Z} = \mathcal{M}^{(0)}_{q\bar{q}\to Z} + \left(\frac{\alpha_s}{4\pi}\right)\mathcal{M}^{(1)}_{q\bar{q}\to Z} + \left(\frac{\alpha_s}{4\pi}\right)^2 \mathcal{M}^{(2)}_{q\bar{q}\to Z} + \dots$$

$$\Rightarrow \left|\mathcal{M}_{q\bar{q}\to Z}\right|^2 = \underbrace{\left|\mathcal{M}^{(0)}_{q\bar{q}\to Z}\right|}_{\text{LO}} + \underbrace{2\,\mathrm{Re}\left[\mathcal{M}^{(0)*}_{q\bar{q}} \times \left(\frac{\alpha_s}{4\pi}\right)\mathcal{M}^{(1)}_{q\bar{q}\to Z}\right]}_{\text{NLO}} + \dots$$

<u>Exercise</u>: compute the 1-loop QCD form factor to $q\bar{q}\to Z$ and show:

$$2\,\mathrm{Re}\left[\mathcal{M}^{(0)*} \times \left(\frac{\alpha_s}{4\pi}\right)\mathcal{M}^{(1)}\right] = \frac{\alpha_s}{2\pi} C_F S_\varepsilon \left(\frac{\mu^2}{m_Z^2}\right)^{\varepsilon}\left[-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \frac{7\pi^2}{6}\right]\left|\mathcal{M}^{(0)}_{q\bar{q}\to Z}\right|^2 \qquad (35)$$

with $S_\varepsilon = (4\pi)^\varepsilon e^{-\gamma_E \varepsilon}$, $\gamma_E \to$ Euler-Mascheroni constant

and $C_F = \frac{N_c^2-1}{2N_c} = \frac{4}{3}$, is the Casimir of the fundamental representation of $SU(N_c)$.

Since the final-state multiplicity is the same as the LO, the phase contribution is equivalent to what we have computed so far. the NLO-virtual contribution to the NLO cross-section is simply proportional to the LO one and reads

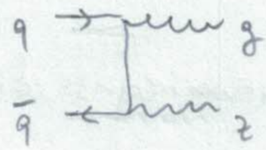
$$d\sigma^{V}_{NLO} = \frac{\alpha_s}{2\pi} C_F S_\varepsilon \left(\frac{\mu^2}{m_Z^2}\right)^{\varepsilon}\left[-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \frac{7\pi^2}{6}\right] d\sigma_{pp\to Z,LO} \qquad (36)$$

*) Cross-section at NLO QCD: real-emission corrections

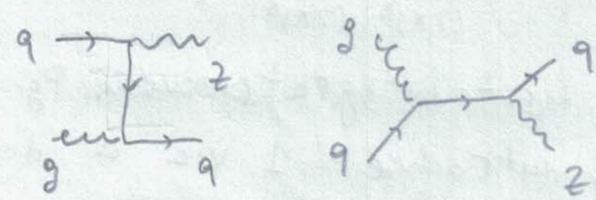
We can define the Drell-Yan process more generally as $pp \to Z + X$, where X is any additional radiation accompanying the Z boson. Since we define our process as Z boson production, X could be $X = \emptyset$ (LO), but also any number of additional QCD interacting particles. At fixed order in perturbation theory the multiplicity of X is limited. At NLO, $\text{mult}(X) \leq 1$, at NNLO $\text{mult}(X) \leq 2$ and so on $\to$ $N^k LO$, $\text{mult}(X) \leq k$.

In our case there are two contributing channels:

1. $q\bar{q} \to Z + g$

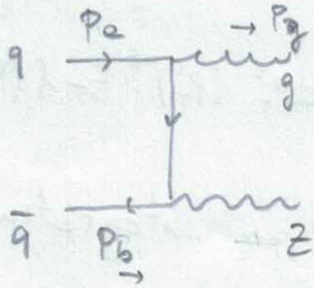


2. $qg \to Z + q$, $gq \to Z + q$
   $(q \leftrightarrow \bar{q})$

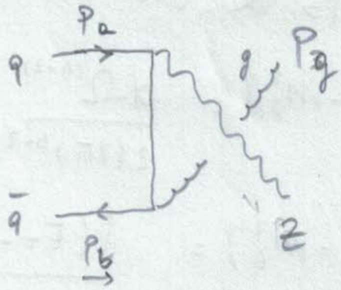


(37)

For now we'll focus on the $q\bar{q}$ channel, and leave the qg one as an exercise. First of all we need to compute the scattering amplitude. We have two contributing diagrams:



t-channel



u-channel

(38)

where we define the kinematics of the process as

$$q(p_a)\ \bar{q}(p_b) \to Z(Q)\ g(p_g) \tag{39}$$

and introduce the following Mandelstam variables:

$$\begin{aligned} &\hat{s} = (p_a + p_b)^2 = x_a x_b S \\ &\hat{t} = (p_a - p_g)^2, \quad \hat{u} = (p_b - p_g)^2 \quad \Rightarrow \quad \hat{s} + \hat{t} + \hat{u} = m_Z^2 \\ &Q^2 = m_Z^2 \end{aligned} \tag{40}$$

Exercise:

compute $|\mathcal{M}_{q\bar{q}\to Zg}|^2$ and show that it is given by

$$|\mathcal{M}_{q\bar{q}\to Zg}|^2 = \underbrace{\left[\frac{4\pi\alpha_W}{N_c}(C_V^2+C_A^2)(1-\varepsilon)m_Z^2\right]}_{|\mathcal{M}^{(0)}_{q\bar{q}\to Z}|^2,\ \text{cfr Eq.}(26)}(4\pi\alpha_s C_F)\times 2\left[\frac{\hat{t}^2+\hat{u}^2+2m_Z^2\hat{s}}{\hat{t}\hat{u}m_Z^2} - \varepsilon\frac{(\hat{u}+\hat{t})^2}{\hat{t}\hat{u}m_Z^2}\right] \quad (41)$$

We see immediately that the matrix element is singular for $\hat{t}\to 0$ or $\hat{u}\to 0$ which corresponds to $p_g\to 0$ or $p_g \parallel p_1$ or $p_g \parallel p_2$.

The phase space in this case reads:

$$d\Phi_{Z+g} = \frac{d^D Q}{(2\pi)^D}\frac{d^D p_g}{(2\pi)^D}(2\pi)^2\delta(Q^2-m_Z^2)\,\delta(p_g^2)\,(2\pi)^D\delta^{(D)}(p_a+p_b-p_g-Q) \quad (42)$$

We first integrate over p_g using the momentum-cons. δ-function and introduce 1 via a δ-function of z $\Rightarrow$

$$d\Phi_{Z+g} = \int_0^1 dz\,\delta\!\left(z-\frac{\tau}{x_a x_b}\right)\frac{d^D Q}{(2\pi)^{D-2}}\,\delta(Q^2-m_Z^2)\,\delta((p_a+p_b-Q)^2) \quad (43)$$

Using for p_a, p_b, Q the parametrisation in Eq. (24) we have

•) $$\frac{d^D Q}{(2\pi)^{D-2}}\delta(Q^2-m_Z^2) = \frac{d\Omega^{(D-2)}}{2(2\pi)^{D-2}}\,dy\,dE_T\,dp_T^2\,(p_T^2)^{-\varepsilon}E_T\,\delta(E_T^2-p_T^2-m_Z^2)$$

$$\delta(E_T^2-p_T^2-m_Z^2) = \frac{\delta(E_T-\sqrt{m_Z^2+p_T^2})}{2\sqrt{m_Z^2+p_T^2}} \Rightarrow \text{integrating over } dE_T$$

$$\Rightarrow \frac{d\Omega^{(D-2)}}{4(2\pi)^{D-2}}\,dy\,dp_T^2\,(p_T^2)^{-\varepsilon}$$

•) $$\delta((p_a+p_b-Q)^2) = \delta(m_Z^2+2p_ap_b-2p_aQ-2p_bQ) =$$

$$= \frac{1}{S}\,\delta\!\left(\tau-\frac{E_T}{\sqrt{S}}(x_a e^{-y}+x_b e^{y})+x_a x_b\right) \quad (44)$$

we then consider the change of variable $p_T^2 \to \lambda$ defined as

$$p_T^2 = \frac{s\tau}{z}\,\bar{z}^2 \lambda \bar{\lambda} \quad \text{with } \bar{z} = 1-z, \quad \bar{\lambda} = 1-\lambda$$

$$dp_T^2 = \frac{s\tau}{z}\,\bar{z}^2 (1-2\lambda)\, d\lambda$$

we can then use the δ-function to constrain the Z-boson rapidity $\Rightarrow$ we get

$$\delta\left(\tau - \frac{E_T}{\sqrt{s}}\left(x_a e^{-y} + x_b e^{y}\right) + x_a x_b\right) = \frac{z}{\tau(1-2\lambda)\bar{z}}\,\delta(y-y^*) \tag{45}$$

$$\text{with } \left[y^* = \frac{1}{2}\log\left(\frac{x_a}{x_b}\right) + \frac{1}{2}\log\left(\frac{1-\bar{z}\bar{\lambda}}{1-\bar{z}\lambda}\right)\right] \tag{46}$$

therefore, back to eq. (43) $\Rightarrow$ (note $z \geq \tau$)

$$d\bar{\Phi}_{Z+g} = \left(\frac{\mu^2}{m_Z^2}\right)^{\varepsilon} \int_\tau^1 dz\, \delta\left(z - \frac{\tau}{x_a x_b}\right) \frac{d\Omega^{(D-2)}}{4(2\pi)^{D-2}}\, dy\, d\lambda\, \delta(y-y^*)\, z^{\varepsilon}\, \bar{z}^{1-2\varepsilon} (\lambda\bar{\lambda})^{-\varepsilon} \tag{47}$$

where we introduced the arbitrary scale μ^2 to have the correct mass-dimensions. Within this parametrization the invariants are

$$\hat{s} = \frac{m_Z^2}{z}, \quad \hat{t} = -\frac{m_Z^2}{z}\,\bar{z}\lambda, \quad \hat{u} = -\frac{m_Z^2}{z}\,\bar{z}\bar{\lambda} \tag{48}$$

and the squared matrix element (+ flux factor)

$$\frac{1}{2\hat{s}}\left|\mathcal{M}_{q\bar{q}\to Z+g}\right|^2 = (4\pi\alpha_s C_F)\,\underset{\substack{\uparrow \\ \text{Eq (26)}}}{\sigma_{LO\varepsilon}}\, \frac{2z}{\pi\tau}\left(\bar{z}^2\lambda\bar{\lambda}\right)^{-1}\left[z + \frac{\bar{z}^2(\lambda^2+\bar{\lambda}^2)}{2} - \frac{\varepsilon\bar{z}^2}{2}\right] \tag{49}$$

Thus the singularity structure in $\bar{z}\to 0$, $\lambda \to 0,1$ is manifest.

First of all we integrate over the angular variables:

$$\int \frac{d\Omega^{(D-2)}}{4(2\pi)^{D-2}} = \frac{1}{8\pi}\frac{(4\pi)^{\varepsilon}}{\Gamma(1-\varepsilon)} = S_\varepsilon \frac{1}{8\pi}\frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)} = \frac{S_\varepsilon}{8\pi}\left[1 - \frac{\pi^2}{12}\varepsilon^2 + \mathcal{O}(\varepsilon^3)\right] \tag{50}$$

$\Rightarrow$ we can put Eq. (47), (49), (50) together and write the real contribution to the hadronic cross-section as:

$$d\sigma^R_{NLO} = \frac{\alpha_s}{2\pi} C_F S_\varepsilon \left(\frac{\mu^2}{m_Z^2}\right)^\varepsilon \sigma_{LO} * \int dx_a\, dx_b\, f_a(x_a) f_b(x_b) \int_\tau^1 dz\, \delta(x_a x_b z - \tau) *$$

$$* \, dy\, \delta(y-y^*) * d\lambda \cdot z^\varepsilon \bar{z}^{-1-2\varepsilon} \bar{\lambda}^{-1-\varepsilon} \lambda^{-1-\varepsilon} * F(z,\lambda) * \frac{\varepsilon^{\gamma_E \varepsilon}}{\Gamma(1-\varepsilon)} \qquad (51)$$

with $F(z,\lambda) = 2z + \bar{z}^2(\lambda^2 + \bar{\lambda}^2) - \bar{z}^2\varepsilon$

There are singularities at $z \to 1$, $\lambda \to 0$ and $\lambda \to 1$ which can be regulated by means of the plus distribution i.e.

$$x^{-1+b\varepsilon} = \frac{\delta(x)}{b\varepsilon} + \sum_{n=0} \frac{(b\varepsilon)^n}{n!} D_n(x), \qquad D_n(x) = \left[\frac{\ln^n(x)}{x}\right]_+ \qquad (52)$$

where $D_n(x)$ is a distribution acting as:

$$\int_0^1 dx\, D_n(x) f(x) = \int^1 dx \frac{\ln(x)}{x} \left[f(x) - f(0) \right] \qquad (53)$$

Second we note we can use a partial fraction identity on λ and write

$$\bar{\lambda}^{-1-\varepsilon} \lambda^{-1-\varepsilon} = \lambda^{-\varepsilon} \bar{\lambda}^{-1-\varepsilon} + \bar{\lambda}^{-\varepsilon} \lambda^{-1-\varepsilon} \qquad (54)$$

so that the $\lambda \to 0$ and $\lambda \to 1$ singularities are approached separately.

Using the plus-distribution on Eq (54) and expanding in $\varepsilon \to 0$ we have:

$$\bar{\lambda}^{-1-\varepsilon} \lambda^{-1-\varepsilon} = -\frac{\delta(\lambda) + \delta(\bar{\lambda})}{\varepsilon} + D_0(\lambda) + D_0(\bar{\lambda}) - \varepsilon\left(D_1(\lambda) + D_1(\bar{\lambda}) + \frac{\ln(\lambda)}{\bar{\lambda}} + \frac{\ln(\bar{\lambda})}{\lambda} \right) + \mathcal{O}(\varepsilon^2) \qquad (55)$$

whereas for $\bar{z}^{-1-2\varepsilon}$ we simply get:

$$\bar{z}^{-1-2\varepsilon} = -\frac{\delta(\bar{z})}{\varepsilon} + D_0(\bar{z}) - 2\varepsilon D_1(\bar{z}) + \mathcal{O}(\varepsilon^2). \qquad (56)$$

Let us combine (55), (56) into (51) and truncate the ε-exp. to ε^{-1}:

$$d\sigma^R_{NLO} = \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{m_Z^2}\right) \sigma_{LO} \int dx_a\, dx_b\, f_a(x_a) f_b(x_b) \int dy\, \delta(y-y^*)\, \delta(x_a x_b z - \tau) *$$

$$dz\, d\lambda \left\{ \frac{\delta(\bar{z})(\delta(\lambda) + \delta(\bar{\lambda}))}{2\varepsilon^2} - \frac{1}{\varepsilon}\left[+ \frac{\delta(\bar{z})}{2}\left(D_0(\lambda) + D_0(\bar{\lambda})\right) + D_1(1-z)(\delta(\lambda) + \delta(\bar{\lambda}))\right]\right\}$$

$$* F(z,\lambda) \qquad (57)$$

$\rightarrow$ note that $\delta(\bar{z}) F(z,\lambda) = 2$

*) ε^{-2} pole $\Rightarrow$ the $\delta(\bar z)$ selects $z=1$ $\Rightarrow$ $y^* = \frac{1}{2}\log\left(\frac{x_a}{x_b}\right) \to$ same as LO

$\to$ anytime $z=1$ we are on a LO configuration $\Rightarrow$

$$d\sigma^R_{NLO}\Big|_{\varepsilon^{-2}} = + \frac{2}{\varepsilon^2}\frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{m_Z^2}\right) d\sigma_{pp\to Z, LO} \tag{56}$$

which cancels exactly the double pole from Eq. (36) (virtual)

*) ε^{-1} pole $\Rightarrow$ one LO-like contribution, $\delta(\bar z)$ and two of type $\delta(\lambda)\,\delta(\bar\lambda)$

We observe that $\delta(\bar z)$ trivializes the dependence on λ as well, indeed $F(z,\lambda)|_{z=1} = 2$ and there's no more λ dependence anywhere.

Therefore any term of type $\delta(\bar z) D_n(\lambda) = \delta(\bar z) D_n(\bar\lambda) = 0 \Rightarrow$

at ε^{-1} we get only: $$\frac{\delta(\lambda)+\delta(\bar\lambda)}{\varepsilon} \times \left(1+z-2D_0(1-z)\right) \tag{57}$$

$\to$ this does not cancel against the virtual contribution.

We can get insights into the meaning of $\lambda \to 0,1$ by studying the kinematics in partonic c.o.m. and parametrise the gluon momentum as

$$p_g = \frac{\sqrt{\hat s}}{2}\,\bar z\,(1, \hat n_g) \Rightarrow \hat t = -2p_1 p_g = -\frac{m_Z^2}{z}\bar z\,\frac{(1-\cos\vartheta)}{2} = -\frac{m_Z^2}{z}\bar z\,\lambda \tag{58}$$

$$\hat u = -2p_2 p_g = -\frac{m_Z^2}{z}\bar z\,\frac{(1+\cos\vartheta)}{2} = -\frac{m_Z^2}{z}\bar z\,\bar\lambda$$

cfr Eq(48)

$\Rightarrow$ $\lambda \to 0$ corresponds to the collinear singularity $p_1 \parallel p_g$

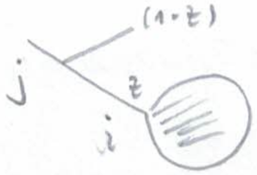
$\lambda \to 1$ " " " " " $p_2 \parallel p_g$

We expect thus that the collinear pole in (57) cancels against collinear PDF renormalization.

*) Cross-section at NLO QCD: collinear PDF renormalization

Where we use the following for an initial-state collinear splitting:

$P_{ij}(z) \longrightarrow$

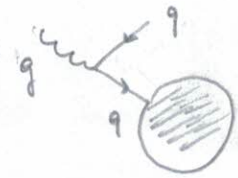


$z \to$ fraction of energy of the initial parton j carried by parton i undergoing the hard scattering

In our Drell-Yan case we need to consider the two type of collinear splittings:

$P_{qq}(z) \longrightarrow$

$P_{qg}(z) \rightarrow$



Their explicit form is:

$$P_{qq}(z) = C_F \left[2 D_0(1-z) - (1+z) + \frac{3}{2}\delta(1-z) \right] \tag{59}$$

$$P_{qg}(z) = T_R \left[z^2 + (1-z)^2 \right]$$

The renormalized PDFs $f_a^{(r)}(x)$ is related to the bare one via:

$$f_a(x) = f_b^{(r)} \otimes \Gamma_{ab} \quad \text{with} \quad (f \otimes g)(x) = \int_0^1 dz_1 dz_2\, f(z_1) g(z_2) \,\delta(x - z_1 z_2) \tag{60}$$

and the kernels Γ_{ab} are given by [in the $\overline{MS}$ scheme]

$$\Gamma_{ab}(x) = \delta_{ab}\,\delta(1-x) + \frac{\alpha_s}{2\pi} \frac{P_{ab}(x)}{\varepsilon} + \mathcal{O}(\alpha_s^2) \,. \tag{61}$$

We need to start from the LO cross-section and renormalize the PDFs there and expand up to $\mathcal{O}(\alpha_s)$. At $\mathcal{O}(\alpha_s^0)$, nothing changes since δ_{ab} preserves flavor and the convolution is trivial.
At $\mathcal{O}(\alpha_s)$ we can start from Eq. (28) and write:

$$\int_\tau^1 dx_a dx_b\, f_q(x_a) f_q(x_b)\, \delta(x_a x_b - \tau) \longrightarrow \text{renormalize } f_a(x_a) \rightarrow$$

$$\frac{1}{\varepsilon} \int_0^1 dz_1 dz_2 \int_\tau^1 dx_a dx_b\, f_q(x_b) * f_q^{(r)}(z_1) P_{qq}(z_2)\, \delta(x_a x_b - \tau)\, \delta(x_a - z_1 z_2) \tag{62}$$

we perform the integration over x_a and simply get:

$$\frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}\int dz_1\, dz_2\, dx_b\, f_q(x_b)\, f^{(r)}(z_1)\, P_{qq}(z_2) \times \delta(z_1 z_2 x_b - \tau)$$

note that $x_a = z_1 z_2 \Rightarrow \tau \le x_a \Rightarrow$ the integration boundaries are

$$\int_\tau^1 dz_1\, dz_2 \to$$ if we rename $z_1 \to x_a$, $z_2 \to z$ we recover exactly the structure of the NLO cross-section.

Thus, after renormalising both $f_a(x_a)$ and $f_b(x_b)$, we get the PDF renormalisation counterterm (we drop the superscript r):

$$\frac{\alpha_s}{2\pi} C_F \frac{1}{\varepsilon}\int_\tau^1 dz\, dx_a\, dx_b\, f_a(x_a)\, f_b(x_b)\, \delta(x_a x_b z - \tau)\Big[\underbrace{4D_0(1-z) - 2(1+z) + 3\delta(\bar z)}_{2\times P_{qq}(z)}\Big] \qquad (63)$$

the contribution coming with a $\delta(1-z)$ cancels against the virtual piece. The other one cancels that from the real-emission contribution (after performing a trivial λ integration).

This completes the IR-pole cancellation. Expanding all the DY cross-section contributions up to $O(\varepsilon^0)$ we finally get

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A) Cross-section at NLO QCD: the $q\bar q$ channel

putting all contributions together we obtain (setting $\mu^2 = m_Z^2$ in the finite contribution): and taking the $\varepsilon \to 0$ limit

$$\sigma_{NLO} = \sigma_{LO}\frac{\alpha_s}{2\pi} C_F \int_\tau^1 dx_a\, dx_b \int_\tau^1 dz \int_0^1 d\lambda \int_{y_{min}}^{y_{max}} dy\, \frac{d^3\hat\sigma}{dy\, dz\, d\lambda} \qquad (64.1)$$

$$\frac{d^3\hat\sigma}{dy\,dz\,d\lambda} = \delta(y-y^*) \times \Big\{ \delta(\bar z)\Big(\pi^2 - 8 + \frac{\ln\bar\lambda}{\lambda} + \frac{\ln\lambda}{\bar\lambda}\Big)\delta(x_a x_b - \tau) +$$

$$+\, \delta(x_a x_b z - \tau) \times \Big[-2\bar z + \big(2D_0(\bar z) - 1 - z\big)\big(D_0(\lambda) + D_0(\bar\lambda)\big) + \qquad (64.2)$$

$$+\, \big(\delta(\lambda) + \delta(\bar\lambda)\big)\Big(\bar z + 4D_1(\bar z) - (1+z)\ln\frac{\bar z^2}{z} - 2\frac{\ln z}{\bar z}\Big)\Big]\Big\}$$

we easily perform the integrations over y and λ to get:

$$\frac{d\sigma}{dz} = \delta(1-z)\,\delta(x_a x_b - \tau)\left(\frac{2\pi^2}{3} - 8\right) + \delta(x_a x_b z - \tau) \times \Big[-2(1+z)\ln\left(\frac{(1-z)^2}{z}\right) + 8 D_1(\bar{z}) - 4\frac{\ln z}{1-z}\Big] \qquad (65)$$

From eq. (64) we can perform the integration over x_a and x_b only and recover the double-differential distribution in y and P_T^2. Recall indeed that $P_T^2 = \frac{m_Z^2}{z}\bar{z}^2\lambda\bar{\lambda}$.

Finally we investigate the behaviour of the cross-section at small values of P_T. Any contribution which features a $\delta(\bar{z})$ is forced onto $P_T^2 = 0$ exactly. Therefore we focus on those w.o. $\delta(\bar{z})$.

Furthermore, we note that the plus distributions regulate the singularity at $P_T^2 = 0$. Since we are approaching $P_T^2 \to 0$, we can remove these prescriptions and select the leading $P_T^2 \to 0$ terms. These reads (we discard $\delta(\lambda)$ and $\delta(\bar{\lambda})$ because lead to $P_T^2 = 0$)

$$\frac{d^2\sigma}{dz\,d\lambda} \underset{P_T^2\to 0}{\simeq} \frac{2}{\bar{z}}\left(\frac{1}{\lambda} + \frac{1}{\bar{\lambda}}\right) \qquad (66)$$

recalling that $\bar{z} = \frac{2P_T}{\sqrt{\hat{s}}}\cosh y_g \geq \frac{2P_T}{\sqrt{\hat{s}}} \underset{P_T\to 0}{\simeq} \frac{2P_T}{m_Z}$ (67)

we can introduce unity as: (68)

$$1 = \int dP_T^2\, \delta\left(P_T^2 - \frac{m_Z^2}{z}\bar{z}^2\lambda\bar{\lambda}\right)$$ → solve the δ wrt λ ⇒

$$\lambda_{1,2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4\frac{P_T^2}{m_Z^2}\frac{z}{\bar{z}^2}}$$ → let's select only one, eg the one $\lambda \to 0$ for $P_T^2 \to 0$, λ_2

obs: for $\lambda = \lambda_2$, $\bar{\lambda} = 1 - \lambda_2 = \lambda_1$ ⇒ in Eq. (66) we get

$$\frac{1}{\lambda_2} + \frac{1}{\bar{\lambda}_2} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \frac{\lambda_1 + \lambda_2}{\lambda_1\lambda_2} = \frac{m_Z^2\bar{z}^2}{P_T^2 z} \qquad (69)$$

upon including the Jacobian from $\delta(P_T^2 - \ldots)$ we get

$$\frac{d\sigma}{dz\,d\lambda\,dp_T^2} = \frac{2}{\bar{z}} \cdot \frac{m_Z^2\bar{z}^2}{P_T^2 z} \cdot \frac{z}{m_Z^2\bar{z}^2} \cdot \frac{1}{1-2\lambda}\,\delta(\lambda - \lambda_2) \qquad (70)$$

we now perform the λ integration, which is trivial, followed by the $\bar{z}$ integration (simply change variable $z \to \bar{z} = 1-z$)

$$\frac{d\sigma}{dp_T^2} \simeq \frac{2}{p_T^2} \int_{2p_T/m_Z} \frac{d\bar{z}}{\bar{z}} \cdot \underbrace{\frac{1}{1-2\lambda_2}}_{\to 1 \text{ for } p_T^2 \to 0} \simeq \frac{2}{p_T^2} \int_{2p_T/m_Z} \frac{d\bar{z}}{\bar{z}} \simeq \frac{1}{p_T^2} \log\left(\frac{m_Z^2}{p_T^2}\right) \tag{70}$$

$\Rightarrow$ which is the well-known logarithmic growth of the p_T^2 distribution towards $p_T^2 \to 0$. Note that we have discarded subleading terms. We can also compute the contribution to the cross-section in the range $p_T^2 \in [p_{T,cut}^2, m_Z^2]$ where $p_{T,cut}^2 \ll m_Z^2$ acts as an infrared cut-off

$$\int_{p_{T,cut}^2}^{m_Z^2} \frac{d\sigma}{dp_T^2} dp_T^2 \simeq \frac{1}{2} \log^2\left(\frac{m_Z^2}{p_{T,cut}^2}\right) \to \text{Leading-logarithmic approximation} \tag{71}$$

This clearly blows up for $p_{T,cut}^2 \to 0$, but the contribution at $p_T^2 \to 0$ from the plus distributions cures this behaviour and ultimately we have a finite cross-section.