Differential Equations

- Recall the notion of an integral family:

$$I(\nu_1, \dots, \nu_L, -\nu_{L+1}, \dots, -\nu_N) = \int_{k_1} \dots \int_{k_l} \frac{S_{L+1}^{\nu_{L+1}} \dots S_N^{\nu_N}}{D_1^{\nu_1} \dots D_L^{\nu_L}}$$

↳ $\nu_i \in \mathbb{Z}$

set of propagators: $\{D_1, \dots, D_L\}$

set of irreducible scalar products: $\{S_{L+1}, \dots, S_N\}$

- any integral in family I is characterized by the list of exponents ν_i
- by construction: any tensor structure containing scalar products of loop momenta and exteral momenta (in accordance with the underlying kinematics) can be absorbed in D_i and S_j

↳ results in linear combination of I at different points $\vec{\nu}$ on the lattice of exponents

↳ can be reduced to master integrals

- Given an integral family I, the number of master integrals is finite (and can be determined with publicly available tools $\begin{cases} \text{Mint [Lee, Pomeransky 2013]} \\ \text{PLD [Fevola, Mizera, Telen 2024]} \end{cases}$)

- Feynman integrals are scalar functions of kinematic invariants and masses

↳ integrals are invariant under Lorentz transformation of external momenta

↳ this property gives rise to the Lorentz-invariance identities and will become important later

- Differential operators

$$O_{ij} = p_i^\mu \frac{\partial}{\partial p_j^\mu} \quad \text{and} \quad m_i^2 \frac{\partial}{\partial m_i^2}$$

are Lorentz invariant and act naturally on the integrand in momentum representation

↳ if m_i is an internal mass (not related to kinematic invariants), $m_i^2 \frac{\partial}{\partial m_i^2}$ will only shift $\nu \to \nu+1$ for all propagators containing m_i

↳ $O_{ij}\, I(\vec{\nu}) = \sum_k C_k\, I(\vec{\nu}_k)$

↳ I at shifted $\vec{\nu}_k$

C_k: rational function depending on kinematic invariants and masses

- Define kinematic invariants:

$$s_{ij} = \begin{cases} p_i \cdot p_j & , \; i=j \\ (p_i+p_j)^2 & , \; i \neq j \end{cases}$$

↳ masses m_i may depend implicitly on momenta

- Integrals only depend on s_{ij} and m_i, but not on the momenta themselves

↳ Goal: express $\frac{\partial}{\partial s_{ij}}$ in terms of O_{ij}

to derive a system of differential equations for the master integrals in the independent invariants

↳ employ the chain rule:

$$O_{ij}\, I = p_i \cdot \frac{\partial I}{\partial p_j} = \sum p_i \cdot \frac{\partial s_{kl}}{\partial p_j} \cdot \frac{\partial I}{\partial s_{kl}}$$

↳ invert the system for $\frac{\partial}{\partial s_{kl}}$

- There are some subtleties:
  - ↳ relations among s_{ij} and m_i, for example, constraint due to Mandelstam relation
  - ↳ operators O_{ij} are not independent

Let us consider an example:

<u>1-Loop Massless Box</u>

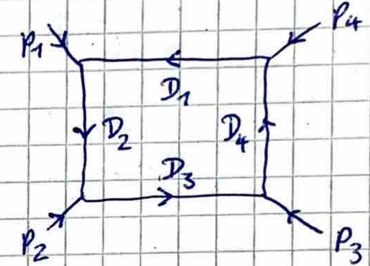
$$I(\nu_1,\nu_2,\nu_3,\nu_4) = \int_k \frac{1}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4}}$$

$D_1 = k^2$

$D_2 = (k+p_1)^2$

$D_3 = (k+p_1+p_2)^2$

$D_4 = (k+p_1+p_2+p_3)^2$



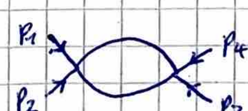
Kinematics: $p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$ $\qquad d = 4-2\varepsilon$

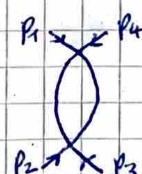
$p_1 + p_2 + p_3 + p_4 = 0$

$s_{12} = 2p_1 \cdot p_2 = s \qquad s_{23} = 2p_2 \cdot p_3 = t \qquad s_{13} = 2p_1 \cdot p_3 = u$

Mandelstam: $s + t + u = 0$

Master integrals:

$I(1,0,1,0)$:  s-channel bubble

$I(0,1,0,1)$:  t-channel bubble

$I(1,1,1,1)$: top level sector

- Treat s,t,u as independent variables and apply chain rule:

$$O_{11} = p_1 \cdot \frac{\partial}{\partial p_1} = p_1 \cdot \left(\frac{\partial s_{12}}{\partial p_1} \frac{\partial}{\partial s_{12}} + \frac{\partial s_{13}}{\partial p_1} \frac{\partial}{\partial s_{13}} + \frac{\partial s_{23}}{\partial p_1} \frac{\partial}{\partial s_{23}} \right)$$

$$= s_{12} \frac{\partial}{\partial s_{12}} + s_{13} \frac{\partial}{\partial s_{13}}$$

$$O_{22} = p_2 \cdot \frac{\partial}{\partial p_2} = s_{12} \frac{\partial}{\partial s_{12}} + s_{23} \frac{\partial}{\partial s_{23}}$$

$$O_{33} = p_3 \cdot \frac{\partial}{\partial p_3} = s_{13} \frac{\partial}{\partial s_{13}} + s_{23} \frac{\partial}{\partial s_{23}}$$

↳ invert the system of equations:

$$s_{12} \frac{\partial}{\partial s_{12}} = \frac{1}{2} \left(O_{11} + O_{22} - O_{33} \right)$$

$$s_{13} \frac{\partial}{\partial s_{13}} = \frac{1}{2} \left(O_{11} - O_{22} + O_{33} \right)$$

$$s_{23} \frac{\partial}{\partial s_{23}} = \frac{1}{2} \left(- O_{11} + O_{22} + O_{33} \right)$$

- Of course, s_{ij} are actually not independent

↳ $s_{12} + s_{13} + s_{23} = 0$

↳ $\frac{\partial}{\partial s_{12}}, \frac{\partial}{\partial s_{13}}, \frac{\partial}{\partial s_{23}}$ cannot be independent

↳ $\frac{\partial}{\partial s} \neq \frac{\partial}{\partial s_{12}}$

How to relate $\left\{ \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right\} \overset{?}{\leftarrow} \left\{ \frac{\partial}{\partial s_{12}}, \frac{\partial}{\partial s_{13}}, \frac{\partial}{\partial s_{23}} \right\}$

u eliminated via $u(s,t) = -s-t$	s_{12}, s_{13}, s_{23} treated as independent variables

- Let $B(s_{12}, s_{23}, s_{13})$ be the integral with independent s_{ij} and $F(s,t)$ the integral with u eliminated.

We identify $F(s,t) = B(s_{12}, s_{23}, -s_{12} - s_{23}) \Big|_{\substack{s_{12}=s \\ s_{23}=t}}$

↳ apply again chain rule to finally obtain the differential operators $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ in terms of $\frac{\partial}{\partial s_{ij}}$ and O_{ij}, respectively:

$$\frac{\partial}{\partial s} F(s,t) = \left(\frac{\partial s_{12}}{\partial s} \frac{\partial}{\partial s_{12}} + \frac{\partial s_{23}}{\partial s} \frac{\partial}{\partial s_{23}} + \frac{\partial s_{13}}{\partial s} \frac{\partial}{\partial s_{13}} \right) B(s_{12}, s_{23}, -s_{12}-s_{23}) \Big|_{\substack{s_{12}=s \\ s_{23}=t}}$$

$$= \left(\frac{\partial}{\partial s_{12}} - \frac{\partial}{\partial s_{13}} \right) B(s,t,-s-t)$$

↳ analogously: $\frac{\partial}{\partial t} = \frac{\partial}{\partial s_{23}} - \frac{\partial}{\partial s_{13}}$

- We can perform another change of variables:

$$\begin{aligned} s &= -\sigma \\ t &= -\sigma x \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \sigma &= -s \\ x &= \frac{t}{s} \end{aligned}$$

↳ thereby, the integral $I(1,1,1,1)$ becomes a function of the dimensionless ratio $x = \frac{t}{s}$ and the dependence on σ can be factored out: $I(1,1,1,1)(s,t) = \sigma^{-2-\varepsilon} \tilde{I}(x)$

(mass)^2-dimension of the integral: $n_e \frac{d}{2} - \sum_{i=1}^{k} \nu_i$

Feynman integrals are homogeneous functions in s_{ij} and m_i

- Let us address the aforementioned issues:

1 : relations among s_{ij} and m_i

↳ In the same manner as intuitively done in the example above, we can obtain relations between the O_{ij} and $\frac{\partial}{\partial s_{ij}}$ by treating the s_{ij} as independent variables.
We take into account possible dependencies between different s_{ij} using the chain rule.

The procedure briefly sketched above can be formulated as a general recipe for deriving differential operators:

- Consider the integral I as a function of independent invariants and masses $x_i \in \{s, t, \dots, m_1^2, \dots\}$.

  ↳ we want to derive the operators $\frac{\partial}{\partial x_i}$

- Consider the "off-shell" integral

$$J = J(\underbrace{\{s_{ij}\}}_{\text{all possible } s_{ij} \text{ as defined previously}}, \underbrace{\{x_i\}}_{\text{only explicit occurances of the parameters, i.e. not implicitly via momenta} \cdot \text{ e.g. propagator masses}})$$

  ↳ for on-shell values of s_{ij}:

$$I = J\Big|_{\text{on-shell}}$$

- The desired differential operators are given by:

$$\frac{\partial I}{\partial x_i} = \left[\underbrace{\frac{\partial s_{kl}}{\partial x_i}}_{= \frac{\partial (s_{kl}|_{\text{on-shell}})}{\partial x_i}} \underbrace{\frac{\partial J}{\partial s_{kl}}}_{} + \frac{\partial J}{\partial x_i} \right]\Bigg|_{\text{on-shell}}$$

↳ compute via

$$O_{ij} I = \underbrace{p_i \cdot \frac{\partial s_{kl}}{\partial p_j}}_{\text{take derivative and replace scalar products by on-shell invariants}} \overbrace{\frac{\partial I}{\partial s_{kl}}}^{\text{solve for}}$$

(note: the system of equations is over-determined)

- this issue will be resolved on the next page
- in practice we can ignore it and solve the system with any IBP reduction software

② : O_{ij} are not independent

↳ In order to relate the operators O_{ij} with the set of differential operators with respect to the invariants, we set up a system of equations using the chain rule.

↳ However, for m independent external momenta there are $m_e \times m_e$ operators O_{ij} and only $\frac{1}{2} m_e(m_e+1)$ kinematic invariants in the off-shell scenario.

Solution: there is a redundancy hidden in the Lorentz-invariance identities!

↳ eliminate the redundancy from the system of equations to render the system invertible

- Recall Lorentz-invariance identities:

$$\sum_{i=1}^{m_e} \left(p_i^\nu \frac{\partial}{\partial p_{i,\mu}} - p_i^\mu \frac{\partial}{\partial p_{i,\nu}} \right) I = 0$$

↳ contract with all anti-symmetric combinations of external momenta $(p_{j,\mu} p_{k,\nu} - p_{k,\mu} p_{j,\nu})$ to get scalar identities

↳ there are $\frac{1}{2} m_e (m_e - 1)$ identities

- Note: We can always add a linear combination of Lorentz-invariance identities to the operators $\frac{\partial}{\partial s_{ij}}$ as obtained with our recipe discussed before.
  Since any Lorentz-invariance identity applied on a scalar integral leads to zero, shifting the $\frac{\partial}{\partial s_{ij}}$ does not affect the final result.

Examples :

- 1 external momentum : $\{p\}$
  ↳ $s_{ij} \in \{s\}$
  ↳ 0 Lorentz-invariance identities

$$0 = p \cdot \frac{\partial}{\partial p} = 2s \frac{\partial}{\partial s} \Rightarrow \frac{\partial}{\partial s} = \frac{1}{2s} p \cdot \frac{\partial}{\partial p}$$

- 2 external momenta : $\{p_1, p_2\}$
  ↳ $s_{ij} \in \{s_{11}, s_{22}, s_{12}\}$
  ↳ 1 Lorentz-invariance identity
  ↳ 4 equations for O_{ij}

$$O_{11} = 2s_{11} \frac{\partial}{\partial s_{11}} + (s_{12} + s_{11} - s_{22}) \frac{\partial}{\partial s_{12}} \qquad (1)$$

$$O_{12} = (s_{12} - s_{11} - s_{22}) \frac{\partial}{\partial s_{22}} + (s_{12} + s_{11} - s_{22}) \frac{\partial}{\partial s_{12}} \qquad (2)$$

$$O_{21} = (s_{12} - s_{11} - s_{22}) \frac{\partial}{\partial s_{11}} + (s_{12} - s_{11} + s_{22}) \frac{\partial}{\partial s_{12}} \qquad (3)$$

$$O_{22} = 2s_{22} \frac{\partial}{\partial s_{22}} + (s_{12} - s_{11} + s_{22}) \frac{\partial}{\partial s_{12}} \qquad (4)$$

↳ system of equations not invertible for $\left\{\frac{\partial}{\partial s_{11}}, \frac{\partial}{\partial s_{22}}, \frac{\partial}{\partial s_{12}}\right\}$

- Lorentz-invariance identity :

$$p^{\mu}_{[1,} p^{\nu}_{2]} \sum_{i=1}^{2} p_{i[\nu,} \frac{\partial}{\partial p_i^{\mu]}}$$

$$= (s_{12} - s_{11} - s_{22})(O_{22} - O_{11}) + 2s_{11} O_{21} - 2s_{22} O_{12} = 0$$

↳ one O_{ij} can be removed from the system above
↳ the system becomes invertible

For example : remove the difference $(O_{22} - O_{11})$
and keep only $\{O_{12}, O_{21}, O_{11} + O_{22}\}$

- 3 external momenta : $\{p_1, p_2, p_3\}$
  ↳ $s_{ij} \in \{s_{11}, s_{22}, s_{33}, s_{12}, s_{13}, s_{23}\}$
  ↳ 3 Lorentz-invariance identities
  ↳ 9 equations for O_{ij}

- In our 1-Loop Massless Box Example we got lucky that due to $p_i^2=0$, all relevant information is encoded in the equations for the operators O_{11}, O_{22}, and O_{33}.
  In principle we could have derived the complete system of equations for all O_{ij} and thereby introduced the operators $\frac{\partial}{\partial s_{ii}}$.
  However, the Lorentz-invariance identities allow to remove 3 equations, leaving 6 equations for 6 $\frac{\partial}{\partial s_{ij}}$.
  The information incorporated in the 3 equations on top of O_{11}, O_{22}, O_{33} can be discarded since $\frac{\partial}{\partial s_{ii}}$ cannot enter the final result.

- 4 external momenta : $\{p_1, p_2, p_3, p_4\}$
  ↳ $\# s_{ij} = 10$
  ↳ 6 Lorentz-invariance identities
  ↳ 16 equations for O_{ij}

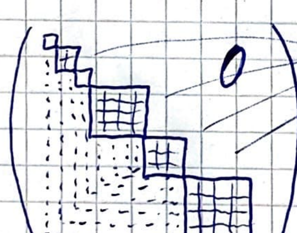
Differential Equations for Master Integrals

- For a given integral family we determine the complete set of master integrals M_i
  - ↳ differentiate all M_i with respect to the kinematic invariants and masses $x_k \in \{s_{ij}, m_i\}$

$$\frac{\partial}{\partial x_k} M_i = A^{(x_k)}_{ij} M_j \rightsquigarrow \frac{\partial}{\partial x_k} \vec{M} = A_k \vec{M}$$

coefficient matrix with rational function coefficients depending on $\{x_i\}$ and dimension d

  - ↳ if we order the master integrals in the vector $\vec{M}$ from the lowest sector to the top level sector, the coefficient matrices will be given in lower block triangular form



every block on the diagonal corresponds to the homogeneous part of the respective sector

- The method of differential equations constitutes a powerful strategy for the computation of master integrals
  - ↳ differential equations comprise information about the analytic structure of the master integrals, e.g. singular points and branch points
  - ↳ asymptotic expansions can systematically be derived to high orders
  - ↳ in contrast to direct integration, e.g. using Feynman parameters, all integrals can be solved "simultaneously" provided the knowledge of a boundary condition

- Properties of Differential Equations:

① : scaling relation:

- Since Feynman integrals are homogeneous functions in the kinematic invariants and masses, the derivatives cannot be independent:

$$I(\lambda^2 s_{ij}, \lambda^2 m_i^2) = \lambda^\alpha I(s_{ij}, m_i^2)$$

$$\Rightarrow \frac{\partial}{\partial\lambda} I(\lambda^2 s_{ij}, \lambda^2 m_i^2) = \alpha \lambda^{\alpha-1} I(s_{ij}, m_i^2)$$

$$\Rightarrow \sum_{kl}\left(s_{kl}\frac{\partial}{\partial s_{kl}} + m_k^2 \frac{\partial}{\partial m_k^2}\right) I(\lambda^2 s_{ij}, \lambda^2 m_i^2) = \frac{\alpha}{2}\lambda^\alpha I(s_{ij}, m_i^2)$$

↳ set $\lambda = 1$ to obtain scaling relation with $\alpha = n_\ell d - 2\sum_{i=1}^{n} \nu_i$

↳ reflects the fact that we have the freedom to set one scale $= 1$ and only consider dimensionless ratios

② : integrability condition:

- according to Schwarz: $\left(\frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial^2}{\partial x_j \partial x_i}\right)\vec{M} = 0$

- apply the product rule to obtain integrability condition:

$$\left(\left(\frac{\partial}{\partial x_j} A_i - \frac{\partial}{\partial x_i} A_j\right) + [A_i, A_j]\right)\vec{M} = 0$$

↳ provides a strong cross-check for the correctness of the IBP reduction

③ : basis transformation:

- the basis of master integral is not unique, i.e. we can apply a rational transformation

↳ $\vec{M}$ satisfies the system $\frac{\partial}{\partial x_k}\vec{M} = A_k \vec{M}$

↳ $\vec{M}' = T^{-1}(\{x_i\}, d)\,\vec{M}$

↳ $\vec{M}'$ satisfies the system $\frac{\partial}{\partial x_k}\vec{M}' = \tilde{A}_k \vec{M}'$ with

$$\tilde{A}_k = T^{-1} A_k T - T^{-1}\frac{\partial}{\partial x_k} T$$