Introduction

These is a collection of lecture notes for a second course in quantum field theory. The notes have grown out of a course on "Advanced Quantum Field Theory", which I have given in the Summer Semesters of 2023 and 2025 at the Technical University of Munich. I do not claim that any of the material presented here is original; on the other hand, the way the theory is developed, especially in the logical order of the various topics, does not follow any standard textbooks. As much as possible, I have tried to to fill in the blanks for various derivations found on standard textbooks.

As it is always the case, these lecture notes cannot be a substitute for one or more textbooks on the subject. Quantum Field Theory is an enormous subject, which remains a source of puzzles and conceptual pitfalls also for the most experts researchers. So my suggestion is, don't give up if things don't seem obvious at first: most of the times, it turns out that they are not! Browse through the many great references available and search for the one that you resonate the most with. Probably, you will need more than one. Here is a (non-exhaustive) list of books that treat the topics covered in this course, sometimes from complementary perspectives.

Standard References

- Maggiore, *A Modern Introduction to Quantum Field Theory* (very short, good for recapping especially QFT 1)
- Peskin, Schröder, *An Introduction to Quantum Field Theory* (a standard reference, we will take a lot from here especially on later chapters)
- Schwartz, *Quantum Field Theory and the Standard Model* (a newer, very good reference, especially for the part on unitarity and for first chapters relevant to QFT 1)
- Weinberg, *The Quantum Theory of Fields, volumes I and II* (the book on QFT: if you have a question, its answer is very probably in one of the Weinberg volumes, but finding it might be part of the adventure...)

For a different "twist" to the topic, and some deep insights

- Itzykson, Zuber, *Quantum Field Theory* (a difficult book, with a lot of interesting material difficult to find anywhere else, for example bound states in QFT)

- Coleman, *Quantum Field Theory* (a beautiful set of lecture notes, with a lot of physics insight. Very long, but worth reading!
- Coleman, *Aspects of Symmetry* (nice self-contained lectures on advanced QFT topics)
- Berestetskii, Lifshitz, Pitaevskii, *Quantum Electrodynamics* (an older book with a lot of profound and half-forgotten results in QED)
- Shifman, *Advanced topics in Quantum Field Theory* (a very advanced book, with interesting special topics)

Chapter 1

Consequences of Unitarity

Quantum Field Theory (QFT) is an extremely rich mathematical framework. In your first encounters with this theory, you have already seen the first glimpses of its richness and complexity. Indeed, as for every physical theory and phenomena, for QFT it is especially true that looking at the same problem from different perspectives can offer unique and sometimes unexpected insights. One of the most striking examples of this, is provided by the recent (re-)discovery of the impact that simple intuitive concepts as those of *Unitarity*, *Causality* and *Locality* can have to constrain the space of available QFTs.

We start this course by reconsidering these concepts and describe how of a large impact they can have in our understanding of Quantum Field Theory. As we will see, following the threads of unitarity and locality and their consequences for the physics of *on-shell* scattering amplitudes, we will be able to justify *a priori* why we are interested in studying so-called *Non-Abelian Yang-Mills* gauge theories, i.e. natural generalizations of QED based on non-abelian *Lie Groups* (in particular, $SU(N)$).

Disclaimer: a lot of the material presented in this chapter has been heavily inspired and adapted from *M. Schwartz, Quantum Field Theory and the Standard Model*, with some obvious reordering, additions and changes.

1.1 The Optical Theorem

The fact that the S-matrix must be unitary is equivalent to the requirement of probability conservation in scattering processes. In fact, consider a system described by a generic Hamiltonian $\mathcal{H}$, such that we can write its S-matrix formally as

$$\mathcal{S} = e^{-i\mathcal{H}t}\,, \tag{1.1}$$

with t the time-evolution variable. Now consider a quantum state in the Schrödinger picture. Its time evolution is given by

$$|\psi; t\rangle = e^{-i\mathcal{H}t}|\psi; 0\rangle\,, \tag{1.2}$$

such that conservation of probability (i.e. that the state remains properly normalized at all times) implies that $\forall\, t$

$$\begin{aligned}\langle\psi;t|\psi;t\rangle &= \langle\psi;0|\psi;0\rangle\,,\\ &\to \langle\psi;0|\mathcal{S}^\dagger\mathcal{S}|\psi;0\rangle = \langle\psi;0|\psi;0\rangle\\ &\to \boxed{\mathcal{S}^\dagger\mathcal{S}=1}\end{aligned} \tag{1.3}$$

i.e. the S matrix must be Unitary. Equivalently, this also implies that

$$\begin{aligned}\left(e^{-i\mathcal{H}t}\right)^\dagger e^{-i\mathcal{H}t} = e^{i\mathcal{H}^\dagger t}e^{-i\mathcal{H}t} = e^{-i(\mathcal{H}-\mathcal{H}^\dagger)t} = 1\\ \to \boxed{\mathcal{H}^\dagger=\mathcal{H}}\end{aligned} \tag{1.4}$$

i.e. $\mathcal{H}$ is Hermitian.

Let us see what the consequences of these simple facts are. We start off by separating the S-matrix into the trivial part (no scattering) and the proper *transfer matrix* T, i.e.

$$\mathcal{S} = 1 + i\,T \tag{1.5}$$

and consider the usual S-matrix elements between an initial state $|i\rangle$ of total four-momentum p_i^μ and a final state $\langle f|$ of total four-momentum p_f^μ

$$\langle f|T|i\rangle = (2\pi)^4\delta^{(4)}(p_i-p_f)\mathcal{M}(i\to f) \tag{1.6}$$

To be 100% clear, these are exactly the matrix elements that we learned to compute with Feynman diagrams in QFT 1!

The formal required that $\mathcal{S}$ is unitary, implies that

$$\begin{aligned}\mathcal{S}^\dagger\mathcal{S} = (1-iT^\dagger)(1+iT) = 1 - i(T^\dagger - T) + T^\dagger T = 1\\ \to \boxed{i(T^\dagger - T) = T^\dagger T}\,.\end{aligned} \tag{1.7}$$

By sandwiching this equation between initial and final state we can write for its left-hand side

$$\begin{aligned}\text{LHS:}\qquad \langle f|i(T^\dagger-T)|i\rangle &= i\left[\langle f|T^\dagger|i\rangle - \langle f|T|i\rangle\right]\\ &= i\left[\langle i|T|f\rangle^* - \langle f|T|i\rangle\right]\\ &= i(2\pi)^4\delta^{(4)}(p_i-p_f)\left[\mathcal{M}^*(f\to i) - \mathcal{M}(i\to f)\right]\,.\end{aligned} \tag{1.8}$$

For the right-hand side instead, we would like to insert a sum over a complete basis of states, which we indicate generically with $|X\rangle$

$$\sum_X \int dX|X\rangle\langle X| = 1\,, \tag{1.9}$$

where we also integrate over all their "quantum numbers" (including their momenta), indicated with the formal integral over dX. For example, if X stands for a set of n final-state scalar particles of momenta p_f and energies E_f, we would have explicitly

$$\int dX = \int \prod_{f=1}^{n} \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\,. \tag{1.10}$$

By inserting this completeness relation, the right-hand-side becomes

$$\begin{aligned}\text{RHS:}\quad \langle f|T^\dagger T|i\rangle &= \sum_X \int dX\, \langle f|T^\dagger|X\rangle\langle X|T|i\rangle \\ &= \sum_X \int dX\, \left[(2\pi)^4\right]^2 \delta^{(4)}(p_i - p_X)\delta^{(4)}(p_f - p_X)\mathcal{M}(i\to X)\mathcal{M}^*(f\to X)\end{aligned} \tag{1.11}$$

and we stress once more that, if the final state particles have other degrees of freedom (spin, color etc), one has to sum over them too. Now we use

$$\delta^{(4)}(p_i - p_X)\delta^{(4)}(p_f - p_X) = \delta^{(4)}(p_f - p_i)\delta^{(4)}(p_i - p_X)$$

and equate LHS and RHS dropping the common $\delta^{(4)}(p_i - p_f)$ to find

$$\boxed{\mathcal{M}(i\to f) - \mathcal{M}^*(f\to i) = i\sum_X \int dX (2\pi)^4 \delta^{(4)}(p_i - p_X)\mathcal{M}(i\to X)\mathcal{M}^*(f\to X)} \tag{1.12}$$

Eq.(1.12) provides the most general form of the *optical theorem*. This is a non-perturbative results, and in fact upon expansion in perturbation theory, the LHS and RHS turn out to connect ℓ-loop quantities on the left, with $(\ell - 1)$ on the right. In fact, imagine, just for definiteness, to work in a scalar theory with coupling constant λ, then obviously since

$$\text{LHS} \quad \sim \mathcal{M}\,, \qquad \text{RHS} \quad \sim |\mathcal{M}|^2$$

eq. (1.12) connects one-loop results on the LHS, to tree-level results on the RHS, etc.

To get the optical theorem in its standard form, one usually considers a specific case where the initial and the final state are identical (i.e. often refereed to as *forward scattering* when dealing with $2\to 2$ scattering processes). In this case $|i\rangle = |f\rangle = |Y\rangle$ and our formula simplifies to

$$\begin{aligned}&\mathcal{M}(Y\to Y) - \mathcal{M}^*(Y\to Y) = \\ &= \boxed{2i\,\text{Im}[\mathcal{M}(Y\to Y)] = i\sum_X \int dX (2\pi)^4 \delta^{(4)}(p_Y - p_X)|\mathcal{M}(Y\to X)|^2}\end{aligned} \tag{1.13}$$

which connects the *imaginary part* of the forward amplitude to the integral of the amplitude squared, for producing all possible intermediate states X. To relate this formula to quantities that we know better and for which we can give an immediate physical interpretation, let us consider two special cases.

1.1.1 Optical theorem for one-particle states

If $|Y\rangle$ is a one-particle state, then we are usually interested in computing its *decay rate*. Indeed, if the state $|Y\rangle$ has a total invariant mass m_Y (it could be for example a Higgs

boson, or any other massive unstable state), the decay rate for $|Y\rangle$ to a specific state $|X\rangle$ is defined as

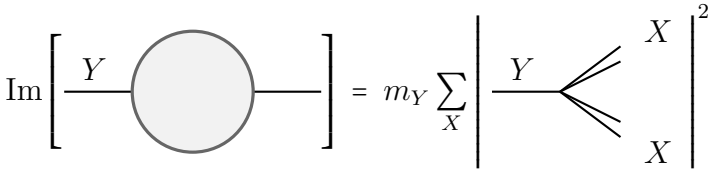
$$\Gamma(Y \to X) = \frac{1}{2m_Y} \int dX (2\pi)^4 \delta^{(4)}(p_Y - p_X) |\mathcal{M}(Y \to X)|^2 \tag{1.14}$$

such that the optical theorem for a one-particle state becomes

$$\boxed{\mathrm{Im}\left[\mathcal{M}(Y \to Y)\right] = m_Y \sum_X \Gamma(Y \to X) = m_Y \Gamma_{\mathrm{tot}}} \tag{1.15}$$

where we indicated with Γ_{tot} the total decay rate of the state Y.

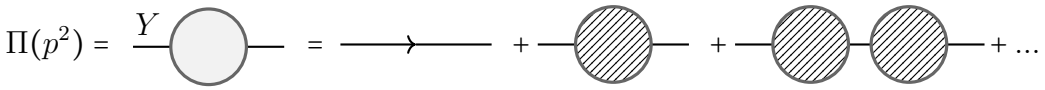
Eq. (1.15) contains the important insight that the two-point correlator for the field corresponding to the state Y, develops an imaginary part if and only if $\Gamma_{\mathrm{tot}} \neq 0$, i.e. there must exist some other state X in the theory for which Y has enough energy to "decay" to. Graphically, we write this equation as



(1.16)

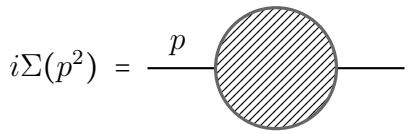
where with the filled bubble we represent the full propagator.

Armed with this result, let us go back and see what we can say about an *unstable* scalar particle Y of invariant mass $p^2 = m_Y^2$. In QFT 1, we have learned to write its propagator, for general values of p^2, by resuming all one-particle irreducible (1PI) contributions as



$$= \frac{i}{p^2 - m_Y^2 + \Sigma(p^2) + i\epsilon}\,, \tag{1.17}$$

where m_Y is the *bare* mass and we defined the self-energy $\Sigma(p^2)$ as the sum of the 1PI graphs as



(1.18)

Now, saying that particle Y is unstable, implies that there is some state X that the particle Y can decay to. As implied by eq. (1.16), the self-energy can then develop an imaginary part if $p^2 > m_X^2$, where m_X^2 is the invariant mass squared of the state X. So for general values of p^2 we must write

$$\Sigma(p^2) = \mathrm{Re}\left[\Sigma(p^2)\right] + i\,\mathrm{Im}\left[\Sigma(p^2)\right]\,. \tag{1.19}$$

With this, the full bare propagator becomes

$$\Pi(p^2) = \frac{i}{p^2 - m_Y^2 + \mathrm{Re}\left[\Sigma(p^2)\right] + i\,\mathrm{Im}\left[\Sigma(p^2)\right]}\,, \tag{1.20}$$

where we remove the $i\epsilon$ prescription, given the imaginary part of the self-energy.

Let us now recall how to define the *renormalized* pole mass. From QFT 1 you might remember that the on-shell (or "pole") mass $\overline{m}_Y^2$ is defined as the pole of the propagator. Whenever the self-energy is real, we can define it by the condition

$$\overline{m}_Y^2 - m_Y^2 + \Sigma(\overline{m}_Y^2) = 0\,. \tag{1.21}$$

This unfortunately cannot be used when the particle is unstable and Σ becomes complex, since this would introduce a complex quantity in our renormalized Lagrangian, which among the other things, would mean a non-hermitian operator! A natural way to define the pole mass (and avoid this problem) in this case, is through the real part of the pole, i.e.

$$\overline{m}_Y^2 - m_Y^2 + \mathrm{Re}\left[\Sigma(\overline{m}_Y^2)\right] = 0\,. \tag{1.22}$$

With this definition for the pole mass, the full renormalized propagator of the unstable particle Y, close to the resonance, i.e. for $p^2 \sim m_Y^2$, becomes

$$\Pi^R(p^2 \sim \overline{m}_Y^2) \;\sim\; \frac{i}{p^2 - \overline{m}_Y^2 + i\,\mathrm{Im}\left[\Sigma(\overline{m}_Y^2)\right]}\,, \tag{1.23}$$

where we have also used the fact that the residue of the propagator at $p^2 = \overline{m}_Y^2$ is normalized to 1 in the on-shell renormalization scheme.

The optical theorem eq. (1.15), allows us to say something more precise about the imaginary part of the propagator:

$$\Gamma_{\mathrm{tot}} = \frac{1}{\overline{m}_Y}\mathrm{Im}\left[\mathcal{M}(Y \to Y)\right] = \frac{1}{\overline{m}_Y}\mathrm{Im}\left[\Sigma(\overline{m}_Y^2)\right] + \mathcal{O}\left(\frac{\Gamma_{tot}}{\overline{m}_Y}\right)\,, \tag{1.24}$$

where the last equality is true modulo non-1PI diagrams. The reason why we wrote a $\mathcal{O}\left(\frac{\Gamma_{tot}}{\overline{m}_Y}\right)$, is that these terms are negligible in first approximation if diagrams the total decay width of the particle is small, which is equivalent to say, if the coupling constant that facilitates this decay is small.[1] With this, we can then finally write, once more close to the resonance,

$$\Pi_R(p^2 \sim \overline{m}_Y^2) \;\sim\; \frac{i}{p^2 - \overline{m}_Y^2 + i\,\overline{m}_Y\Gamma_{\mathrm{tot}}}\,, \tag{1.25}$$

which is called a *Breit-Wigner propagator*.

Imagine now that this particle is exchanged in some s-channel scattering process $p_1+p_2 \to Y \to p_3+p_4$. The cross-section for this process will be proportional to the amplitude squared. If we assume that the particle Y has a very small width, i.e. *the resonance is very narrow*, then we can with good precision approximate

$$\mathrm{Im}\left[\Sigma(s)\right] \sim \mathrm{Im}\left[\Sigma(\overline{m}_Y^2)\right]$$

[1] An equivalent way to see this, is using the fact that the scattering amplitude for $Y \to Y$ is, through the LSZ formula $\mathcal{M}(Y \to Y) \propto Z\Sigma(\overline{m}_Y)$, and $Z = 1 + \mathcal{O}(\lambda)$, for any small coupling λ in the theory.

over the whole width of the resonance, which is also where most of the cross-section will come from. In this way, we get for the cross-section close to the resonance, in very good approximation

$$\sigma \propto \left| \begin{array}{ccc} p_1 & & p_3 \\ & Y & \\ p_2 & & p_4 \end{array} \right|^2 \propto \left| \frac{i}{p^2 - \overline{m}_Y^2 + i\overline{m}_Y \Gamma_{\text{tot}}} \right|^2 = \frac{1}{(p^2 - \overline{m}_Y^2)^2 + (\overline{m}_Y \Gamma_{\text{tot}})^2} \tag{1.26}$$

which is true, once more, as long as $\Gamma_{\text{tot}} \ll m_Y$. The last form of eq. (1.26) is called the Breit-Wigner distribution, and it is the tipical shape of resonances observed at particle colliders, see fig **??**.[ADD FIGURE]

1.1.2 Optical theorem for two-particle states

Consider now the case of a two-particle state, schematically $|Y\rangle = |p_1 p_2\rangle$. In this case, the cross section for producing any state $|X\rangle$ in the scattering process $p_1 + p_2 \to X$, with center of mass energy $(p_1 + p_2)^2 = s$, is defined as

$$\sigma(p_1 p_2 \to X) = \frac{1}{4\sqrt{s}|\vec{p}_{cm}|} \int dX (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_X) |\mathcal{M}(p_1 p_2 \to X)|^2 \tag{1.27}$$

where $\vec{p}_{cm}$ is the three-momentum of one of the two incoming particles in the center-of-mass frame (clearly, which one does not matter, since we only need its modulus squared). In this case, the optical theorem becomes

$$\text{Im}\left[\mathcal{M}(p_1 p_2 \to p_1 p_2)\right] = 2\sqrt{s}|\vec{p}_{cm}| \sum_X \sigma(p_1 p_2 \to X)\,, \tag{1.28}$$

i.e. the imaginary part of the *forward* $2 \to 2$ *scattering amplitude* is equal to the total cross section for $p_1 + p_2 \to$ anything. Often, when people talk about the optical theorem, they refer to this special case for two-particle scattering.

1.2 Imaginary parts from discontinuities

In the previous section, we have discussed how the optical theorem connects the "imaginary part" of forward scattering amplitudes, to cross-sections (or decay rates). We have also seen that the optical theorem, applied to a single particle state, suggests that imaginary parts are generated as long as there is enough energy for that particle to decay to some other state. In this section, we would like to investigate more in detail where these imaginary parts come from, first in an explicit example, and then introducing the so-called "Cutkosky's cutting rules", which allow to derive the imaginary part of an amplitude, or an integral, but computing their "unitarity cuts".

1.2.1 An explicit calculation of an imaginary part

Let us start with an example. We work in the usual theory of a real scalar field with Lagrangian

$$\mathcal{L}_{\rm int} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

and consider $2 \to 2$ scattering process:

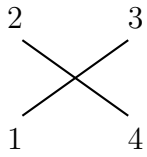
$$\phi(p_1) + \phi(p_2) \to \phi(p_3) + \phi(p_4)\,. \tag{1.29}$$

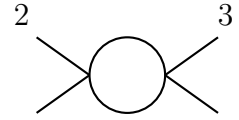
The four momenta are on the mass-shell, i.e. $p_i^2 = m^2$ for all $i = 1, ..., 4$ and we introduce the usual Mandelstam variables to parametrize the kinematics

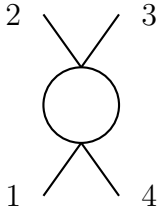
$$s = (p_1 + p_2)^2\,, \quad t = (p_1 - p_3)^2\,, \quad u = (p_2 - p_3)^2\,, \quad \text{with} \quad s + t + u = 4m^2\,. \tag{1.30}$$

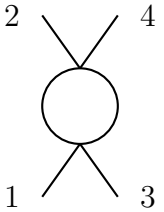
In the physical scattering region, one needs $s > 4m^2$, i.e. s must be *above threshold* to produce the two particles in the final state. The kinematical constrains above tell us that $t + u < 0$, and it is easy to prove that individually both $t, u < 0$.

Let us consider now this calculation up to $\mathcal{O}(\lambda^2)$, or one-loop order. The relevant Feynman diagrams are

$i\mathcal{M}^{1\ell}(12 \to 34) =$ 

$+\frac{1}{2}\Bigg[$ 

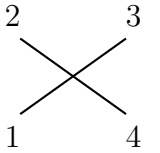
$+$ 

$+$ 

$\Bigg] + \mathcal{O}(\lambda^2)$

(1.31)

where the only relevant Feynman rule is



$= -i\lambda\,,$

which also corresponds to the tree-level four-point amplitude above.

We would like to discuss which of these diagrams can produce an imaginary part in the kinematical situation described by eq. (1.30). We start analyzing explicitly the s-channel

one-loop diagram. Writing for compactness $p_1 + p_2 = p$, with $p^2 = s$, we have explicitly

$$i\mathcal{M}_s(p^2) = \quad \begin{matrix} 2 & & 3 \\ & \bigcirc & \\ 1 & & 4 \end{matrix}$$

$$= -\lambda^2 \int \frac{d^D k}{(2\pi)^D} \frac{i^2}{(k^2 - m^2 + i\delta)((k+p)^2 - m^2 + i\delta)} \tag{1.32}$$

where we used δ for the Feynman prescription, to avoid confusion with the D-dimensional regulator $D = 4 - 2\epsilon$. Introducing Feynman parameters and completing the square as usual, we can write

$$\begin{aligned} \frac{1}{(k^2 - m^2 + i\delta)((k+p)^2 - m^2 + i\delta)} &= \int_0^1 dx \frac{1}{\left[(k^2 - m^2)(1-x) + (k^2 + p^2 + 2k\cdot p)x - m^2 x + i\delta\right]^2} \\ &= \int_0^2 dx \frac{1}{\left[(k+px)^2 + p^2 x(1-x) - m^2 + i\delta\right]^2} . \end{aligned} \tag{1.33}$$

We then put this back in the original $d^D k$ integral and shift $k \to k - px$ to obtain

$$\begin{aligned} i\mathcal{M}_s(p^2) &= \lambda^2 \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left[k^2 + p^2 x(1-x) - m^2 + i\delta\right]} \\ (\textit{Wick Rotation}) \;\to\; &= i\lambda^2 \int_0^1 dx \int \frac{d^D k_E}{(2\pi)^D} \frac{1}{\left[k_E^2 + p_E^2 x(1-x) + m^2\right]} \\ (\textit{with } p_E^2 = -p^2 - i\delta) \;\to\; &= i\lambda^2 \frac{\Gamma(\epsilon)}{(4\pi)^{D/2}} \int_0^1 dx \left[p_E^2 x(1-x) + m^2\right]^{-\epsilon} . \end{aligned} \tag{1.34}$$

Now we use $\Gamma(1+\epsilon) = \epsilon\Gamma(\epsilon)$ to extract explicitly the pole and expand the integrand close to $\epsilon = 0$ to obtain

$$i\mathcal{M}_s(p^2) = \frac{i\lambda^2}{(4\pi)^{D/2}} \Gamma(1+\epsilon) \left[\frac{1}{\epsilon} - \int_0^1 dx \log\left(m^2 + p_E^2 x(1-x)\right)\right] + \mathcal{O}(\epsilon) . \tag{1.35}$$

From inspecting the result above, we see that the logarithm is real as long as $m^2, p_E^2 > 0$. For physical values of the external kinematics, though, as we know $p_E^2 = -p^2 - i\delta < 0$ and the logarithm can develop an imaginary part. In particular, the argument of the logarithm can become zero for two values of $x_\pm$

$$m^2 - p^2 x(1-x) = 0 \quad \to \quad x_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{4m^2}{p^2}} . \tag{1.36}$$

From this, it is clear that the logarithm behaves differently depending on the values of p^2.

- For $p^2 < 4m^2$, there is no real solution for $0 < x < 1$, which means that we can integrate the logarithm and we will obtain a real result.
- If instead $p^2 > 4m^2$, one finds that both solutions are real and, in particular, $0 < x_\pm < 1$, such that they both lie on the integration contour!

It is here that we need Feynman's prescription $i\delta$ to specify on which side of the branch cut we are. In particular, using the fact that $x(1-x) > 0$, we can reinstate the Feynman prescription directly in the logarithm. We also define $\Delta(x) = m^2 - p^2 x(1-x)$ and notice that

$$\Delta(x) > 0\,, \qquad \text{for} \quad 0 < x < x_- \;\&\; x_+ < x < 1\,, \tag{1.37}$$

$$\Delta(x) < 0\,, \qquad \text{for} \quad x_- < x < x_+\,. \tag{1.38}$$

Let us then split the integration contour and write *(very explicitly...)*

$$\begin{aligned}\int_0^1 dx \log\left(m^2 - p^2 x(1-x) - i\delta\right) &= \int_0^{x_-} dx \log\left(\Delta(x)\right) + \int_{x_+}^1 dx \log\left(\Delta(x)\right) \\ &+ \int_{x_-}^{x_+} \log\left(-[-\Delta(x)] - i\delta\right) \\ &= \left[\int_0^{x_-} + \int_{x_+}^1\right] dx \log\left(\Delta(x)\right) \\ &+ \int_{x_-}^{x_+} dx \log\left([-\Delta(x)]\right) + \log\left(-1 - i\delta\right) \int_{x_-}^{x_+} dx\,. \end{aligned} \tag{1.39}$$

Using $\log(-1 - i\delta) = -i\pi$ we finally can extract the imaginary part of this integral as

$$\mathrm{Im}\left[\int_0^1 dx \log\left(m^2 - p^2 x(1-x) - i\delta\right)\right] = -i\pi\sqrt{1 - \frac{4m^2}{p^2}}\,\theta(p^2 - 4m^2)\,, \tag{1.40}$$

where we introduced an explicit θ-function to stress that the imaginary part is different from zero only if $p^2 > 4m^2$. Putting everything together, this finally gives for the amplitude in the s channel (and writing for consistency with the usual Mandelstam variables notation $p^2 = s$)

$$i\mathcal{M}_s(s) = \frac{i\lambda^2}{(4\pi)^{D/2}}\Gamma(1+\epsilon)\left[\frac{1}{\epsilon} - \mathrm{Re}\left(\int_0^1 dx \log\left(\Delta(x)\right)\right) + i\pi\sqrt{1 - \frac{4m^2}{s}}\,\theta(s - 4m^2)\right]. \tag{1.41}$$

Since the imaginary part is finite in $D = 4$, we can write (removing the overall i factor)

$$\mathrm{Im}\left[\mathcal{M}_s(s)\right] = \frac{\lambda^2}{16\pi^2}\sqrt{1 - \frac{4m^2}{s}}\,\theta(s - 4m^2) \tag{1.42}$$

NOTE: Often, people talk interchangeably about *imaginary part* or *discontinuity* of an amplitude. While this is not entirely precise, we will use this nomenclature as well, meaning typically the imaginary part developed by an amplitude in one single variable. Indeed, we define the discontinuity of the amplitude in the variable s as the difference of its value across the branch-cut for $s > 4m^2$:

$$\mathrm{Disc}\left(\mathcal{M}_s(s)\right) = \lim_{\delta\to 0}\left[\mathcal{M}_s(s + i\delta) - \mathcal{M}_s(s - i\delta)\right] = 2i\,\mathrm{Im}\left[\mathcal{M}_s(s)\right]\,. \tag{1.43}$$

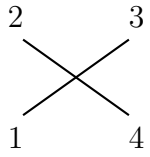
Let's go back now to the full amplitude, eq. (1.31). We analyzed the first of three one-loop graphs and discovered that it develops an imaginary part (or a discontinuity) whenever $s > 4m^2$. What about the other graphs? Is there any other contribution to the imaginary part of the amplitude? Clearly, if we were to compute the t- and u-channel graphs, the would give exactly the same result as the s-channel one, with $s \leftrightarrow t, u$. This also means, that those diagrams will develop an imaginary part if and only if $t > 4m^2$ and $u > 4m^2$ respectively. But as we discussed above, in the scattering region $12 \to 34$, only $s > 4m^2$, while t, u are negative, so the only contribution to the imaginary part of the full amplitude is the one that we computed in (1.42). Putting this back into the equation for the amplitude eq. (1.31), we get (putting back in the symmetry factor 1/2)

$$\begin{aligned} \mathrm{Im}\left[\mathcal{M}^{1\ell}(12\to 34)\right] &= \frac{\lambda^2}{32\pi}\sqrt{1-\frac{4m^2}{s}}\theta(s-4m^2)\,. \\ (\textit{forward limit}) \;\to \mathrm{Im}\left[\mathcal{M}^{1loop}(12\to 12)\right] &= \frac{\lambda^2}{32\pi}\sqrt{1-\frac{4m^2}{s}}\theta(s-4m^2) \end{aligned} \tag{1.44}$$

where in the last line we noticed that in the forward limit nothing changes, since the imaginary part does not depend on p_3 or p_4, but only on s!

Comparing with the Optical theorem

We can now verify the validity of the optical theorem *perturbatively* to $\mathcal{O}(\lambda^2)$. In this case, it implies that the imaginary part we just computed should be equal to the cross section for $12 \to X$. To this order in perturbation theory, the only possible process contributing is the tree-level diagram with $X = 34$:



$$= i\mathcal{M}^{tree}(12\to 34) = -i\lambda \quad \to \quad |\mathcal{M}^{tree}(12\to 34)|^2 = \lambda^2\,.$$

The cross-section then becomes

$$\sigma(12\to 34) = \frac{1}{4\sqrt{s}\,|\vec{p}_{cm}|}\underbrace{\frac{1}{2!}}_{\textit{identical particles}}\int \frac{d^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{d^3\vec{p}_4}{(2\pi)^3 2E_4}(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\lambda^2$$

$$= \frac{1}{2\sqrt{s}\,|\vec{p}_{cm}|}\frac{\lambda^2}{4}\,\Pi_2 \tag{1.45}$$

where we introduced the notation for the two-particle phase-space

$$\begin{aligned}\Pi_2 &= \int \frac{d^3\vec{p}_3}{(2\pi)^3 2E_3}\frac{d^3\vec{p}_4}{(2\pi)^3 2E_4}(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\\ &= \int \frac{d^4p_3}{(2\pi)^4}(2\pi)\delta^+(p_3^2-m^2)\int \frac{d^4p_4}{(2\pi)^4}(2\pi)\delta^+(p_4^2-m^2)\,(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\,,\end{aligned} \tag{1.46}$$

with, given a four-vector x^μ, the usual definition $\delta^+(x^2)=\delta(x^2)\theta(x_0)$.

The computation of the phase-space is standard. We integrate out the four-dimensional delta function removing the momentum p_4, and then specify the calculation in the center-of-mass frame of p_1+p_2, i.e. choose $p_1+p_2=(\sqrt{s},\vec{0})$, to find

$$\begin{aligned}\Pi_2 &= \int \frac{d^4p_3}{(2\pi)^4}(2\pi)^2\delta(p_3^2-m^2)\theta(E_3)\theta(E_1+E_2-E_3)\delta((p_1+p_2-p_3)^2-m^2)\\ (c.o.m\ frame)\ \rightarrow &= \int \frac{d^4p_3}{(2\pi)^2}\delta(p_3^2-m^2)\theta(E_3)\theta(\sqrt{s}-E_3)\delta(s-2\sqrt{s}E_3)\\ &= \frac{1}{2\sqrt{s}}\int\frac{d^3\vec{p}_3}{(2\pi)^2}\int_0^{\sqrt{s}}dE_3\,\delta(E_3^2-\vec{p}_3^2-m^2)\,\delta(E_3-\sqrt{s}/2)\\ &= \frac{1}{2\sqrt{s}}\int\frac{d^3\vec{p}_3}{(2\pi)^2}\delta\left(\frac{s}{4}-m^2-|\vec{p}_3|^2\right)\\ (spherical\ coord.)\ \rightarrow &= \frac{1}{2\sqrt{s}}\frac{4\pi}{4\pi^2}\int_0^\infty d|\vec{p}_3|\,|\vec{p}_3|^2\delta\left(\left(|\vec{p}_3|-\sqrt{\frac{s}{4}-m^2}\right)\left(|\vec{p}_3|+\sqrt{\frac{s}{4}-m^2}\right)\right)\\ &= \frac{1}{2\sqrt{s}\pi}\left(\frac{s}{4}-m^2\right)\frac{1}{2\sqrt{\frac{s}{4}-m^2}} = \frac{1}{4\pi\sqrt{s}}\sqrt{\frac{s}{4}-m^2}\,.\end{aligned} \tag{1.47}$$

Putting everything together, we can write for the two-particle phase space

$$\boxed{\Pi_2 = \frac{1}{8\pi}\sqrt{1-\frac{4m^2}{s}}}\,, \tag{1.48}$$

such that the cross section in eq. (1.45) reads

$$\sigma(12\to 34) = \frac{1}{2\sqrt{s}|\vec{p}_{cm}|}\left(\frac{\lambda^2}{4}\right)\frac{1}{8\pi}\sqrt{1-\frac{4m^2}{s}}\theta(s-4m^2)\,, \tag{1.49}$$

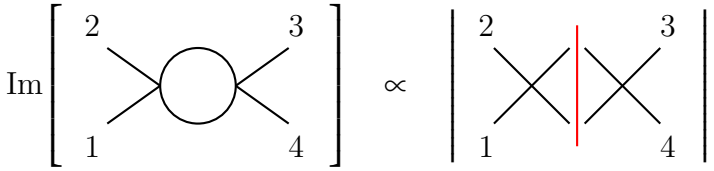
where we added a $\theta(s-4m^2)$ to stress that the cross-section is different from zero only if enough energy is available.

We can now use the cross section just computed in eq. (1.49) to verify the optical theorem to order $\mathcal{O}(\lambda^2)$. In fact, eq. (1.28) to this order implies

$$\begin{aligned}\mathrm{Im}\left[\mathcal{M}(12\to 12)\right] &= 2\sqrt{s}|\vec{p}_{cm}|\sum_X \sigma(12\to X)\\ (to\ order\ \mathcal{O}(\lambda^2))\ \rightarrow &= 2\sqrt{s}|\vec{p}_{cm}|\,\sigma(12\to 34)\\ &= \frac{\lambda^2}{32\pi^2}\sqrt{1-\frac{4m^2}{s}}\,,\end{aligned} \tag{1.50}$$

which indeed agrees with the imaginary part computed in eq. (1.44). We notice that we never needed the explicit value of $|\vec{p}_{cm}|$, since it canceled between the expression for the imaginary part in terms of the cross-section, and the cross-section computed in eq. (1.49).

In conclusion, we have proved the optical theorem in general and verified its validity with an explicit calculation. In doing so, we have seen that the one-loop amplitude in eq. (1.31) develops an imaginary part when $s > 4m^2$. As we will demonstrate in the next section, this is the *threshold* energy necessary for the virtual particles exchanged in the loop to get on-shell *(it has nothing to do with the energy required to produce the final state particles!)*. Graphically, this is often expressed as a "cut" through the virtual particles that have to be produced on-shell:



(1.51)

1.2.2 Cutkosky's cutting rules

We have seen an explicit calculation that suggests that "discontinuities" (or imaginary parts) of scattering amplitudes seem to be related to kinematical configurations where virtual particles can be produced on-shell. In this section, we will make this statement more precise and introduce a set of rules to compute discontinuities without having to first compute the full Feynman diagram, as we did in the previous lecture.

Let us consider some generic scattering amplitude in some scalar theory, involving N external particles, say

$$p_1 + p_2 \to p_3 + \ldots + p_N \,.$$

The integrand for such an amplitude, is always a combination of Feynman propagators

$$\Pi_F(k) = \frac{i}{k^2 - m^2 + i\epsilon} \,. \tag{1.52}$$

NOTE: We are considering a scalar theory for simplicity, but all our considerations extend more or less straightforwardly to any spin. On way to convince yourself that this is true, is realizing that an amplitude in a theory with particles with spin, can always be rewritten, after some trivial algebraic manipulations, as a combination of purely scalar Feynman integrals, which are also the only source of possible discontinuities or branch cuts. You have already seen many examples of this, for example when you computed the electron self-energy in QED or the g-2 of the electron. In the case of the self-energy, you probably wrote it as something like

$$\hat{\Sigma}(p, m) = \Sigma_V(p^2, m^2)\, \not{p} + \Sigma_S(p^2, m^2)\, m\mathbb{I} \,, \tag{1.53}$$

where the entire spinor-structure is contained in the matrices $\not{p}$ and $\mathbb{I}$, while Σ_V and Σ_S are scalar functions, containing Feynman integrals with propagators exactly in the form of (1.52).

Since we are interested in obtaining the discontinuity of an amplitude, we start, with some abuse of language, defining the "discontinuity" of an individual Feynman propagator

$$\text{Disc}\left[\Pi_F(k)\right] = \lim_{\epsilon\to 0}\left[\frac{i}{k^2-m^2+i\epsilon} - \frac{i}{k^2-m^2-i\epsilon}\right] = \lim_{\epsilon\to 0}\frac{2\epsilon}{(k^2-m^2)^2+\epsilon^2}\,. \tag{1.54}$$

Here, we are using the fact that all branch cuts originate from poles in the propagators and, in turn, the imaginary parts in the Feynman amplitudes are fixed by Feynman's prescription. In this sense, computing the discontinuity of a graph is related to the value of the same graph with flipped signs of the Feynman $i\epsilon$, which justifies the terminology.

In the limit for $\epsilon \to 0$ we can recognize the famous representation of the Dirac delta function

$$\lim_{\epsilon\to 0}\left[\frac{\epsilon}{x^2+\epsilon^2}\right] = \pi\delta(x) \tag{1.55}$$

which has to be interpreted as an identity at the distribution level, i.e. for our purposes valid in the integral sign

$$\lim_{\epsilon\to 0}\left[\epsilon\int_{-\infty}^{\infty}\frac{dx}{x^2+\epsilon^2}f(x)\right] = \pi f(0)\,. \tag{1.56}$$

With this, we can identify (again at the distribution level)

$$\text{Disc}\left[\Pi_F(k)\right] = 2i\,\text{Im}\left[\Pi_F(k)\right] = 2\pi\,\delta(k^2-m^2)\,, \tag{1.57}$$

which shows that propagators can contribute to a discontinuity only when they go on shell.

Let's go back to the Feynman propagator now, and split its energy component and partial fraction it to write

$$\Pi_F(k) = \frac{i}{k_0^2-\omega_k^2+i\epsilon} = \frac{i}{2\omega_k}\left[\frac{1}{k_0-\omega_k+i\epsilon} - \frac{1}{k_0+\omega_k-i\epsilon}\right] \tag{1.58}$$

with $\omega_k = \sqrt{\vec{k}^2+m^2}$. Let's now add and subtract from this the complex conjugate of the first term, i.e.

$$\Pi_F(k) = \underbrace{\frac{i}{2\omega_k}\left[\frac{1}{k_0-\omega_k-i\epsilon} - \frac{1}{k_0+\omega_k-i\epsilon}\right]}_{\text{retarded propagator }\Pi_R(k)} + \underbrace{\frac{i}{2\omega_k}\left[\frac{1}{k_0-\omega_k+i\epsilon} - \frac{1}{k_0-\omega_k-i\epsilon}\right]}_{\frac{-2i\epsilon}{(k_0-\omega_k)^2+\epsilon^2}} \tag{1.59}$$

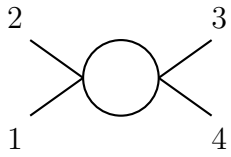
such that we can write

$$\begin{aligned}\Pi_F(k) &= \Pi_R(k) + \frac{1}{\omega_k}\frac{\epsilon}{(k_0-\omega_k)^2+\epsilon^2}\\ &= \Pi_R(k) + \frac{\pi}{\omega_k}\delta(k_0-\omega_k) \quad (\textit{at the distribution level})\,,\end{aligned} \tag{1.60}$$

and we have introduced the retarded propagator, which has the peculiarity of having poles only in the upper half complex plane (i.e. above the real axis).

Let's now consider the one-loop diagram studied in the previous section eq. (1.32), which for the current argument we rewrite as

$$i\mathcal{M}_s(s) = \quad = -\lambda^2 \int \frac{d^D k}{(2\pi)^D} \Pi_F(k)\Pi_F(k-p)\,. \tag{1.61}$$



We now use eq. (1.60) and do a bit of gymnastic, which might look a bit funny at first, to find

$$\begin{aligned} i\mathcal{M}_s(s) = -\lambda^2 \Bigg[& \int \frac{d^D k}{(2\pi)^D} \Pi_R(k)\Pi_R(k-p) \\ & + \int \frac{d^D k}{(2\pi)^D} \frac{\pi^2}{\omega_k \omega_{k-p}} \delta(k_0 - \omega_k)\delta(k_0 - p_0 - \omega_{k-p}) \\ & + \int \frac{d^D k}{(2\pi)^D} \frac{\pi}{\omega_k} \delta(k_0 - \omega_k)\Pi_R(k-p) \\ & + \int \frac{d^D k}{(2\pi)^D} \frac{\pi}{\omega_{k-p}} \delta(k_0 - p_0 - \omega_{k-p})\Pi_R(k) \Bigg]. \end{aligned} \tag{1.62}$$

Of the four terms above, one can easily see that the first two are identically zero. In particular:

$$I_1 = \int \frac{d^D k}{(2\pi)^D} \Pi_R(k)\Pi_R(k-p) = 0 \tag{1.63}$$

because when doing the energy integral (in k_0), one realizes that the integrand has only poles above the real axis, so closing the contour below the real axis gives identically zero. Let us consider the second integral instead, and for to the center of mass frame for $p = (\sqrt{s}, \vec{0})$. In this frame $\omega_k = \omega_{k-p}$ and we obtain

$$\begin{aligned} I_2 &= \int \frac{d^D k}{(2\pi)^D} \frac{\pi^2}{\omega_k \omega_{k-p}} \delta(k_0 - \omega_k)\delta(k_0 - p_0 - \omega_{k-p}) \\ &= \int \frac{d^D k}{(2\pi)^D} \frac{\pi^2}{\omega_k^2} \delta(k_0 - \omega_k)\delta(k_0 - \sqrt{s} - \omega_k) = 0 \end{aligned} \tag{1.64}$$

because $s > 4m^2$ and the two delta functions would require $\sqrt{s} = 0$. The last two integrals don't drop naively, so let us manipulate them "back" into a more standard form, by re-inserting the inverse of eq. (1.60) to reintroduce the Feynman propagator. When doing that, we will obtain again two terms with products of two delta functions as I_2 above, which we know can be dropped for the same reason. The terms that remain are then the

last two, with the retarded propagator substituted by the Feynman one

$$i\mathcal{M}_s(s) = -\lambda^2\Bigg[\int \frac{d^Dk}{(2\pi)^D}\frac{\pi}{\omega_k}\delta(k_0-\omega_k)\Pi_F(k-p) + \int \frac{d^Dk}{(2\pi)^D}\frac{\pi}{\omega_{k-p}}\delta(k_0-p_0-\omega_{k-p})\Pi_F(k)\Bigg]. \tag{1.65}$$

This expression is exact, in all our manipulations we never threw away anything.

We would like now to use this expression to compute the imaginary part of $\mathcal{M}_s$. Indeed, in this form it is easy to do wo, because the delta functions are clearly real (the only contain energies), the only possible sources of discontinuities (or branch cuts) are the Feynman propagators. Using eq. (1.57) we find

$$\begin{aligned}\text{Disc}\left[i\mathcal{M}_s(s)\right] &= -\lambda^2\Bigg[\int \frac{d^Dk}{(2\pi)^D}\frac{\pi}{\omega_k}\delta(k_0-\omega_k)\text{Disc}\left[\Pi_F(k-p)\right] \\ &\qquad + \int \frac{d^Dk}{(2\pi)^D}\frac{\pi}{\omega_{k-p}}\delta(k_0-p_0-\omega_{k-p})\text{Disc}\left[\Pi_F(k)\right]\Bigg] \\ &= -\lambda^2\Bigg[\int \frac{d^Dk}{(2\pi)^D}\frac{\pi}{\omega_k}\delta(k_0-\omega_k)2\pi\delta((k-p)^2-m^2) \\ &\qquad + \int \frac{d^Dk}{(2\pi)^D}\frac{\pi}{\omega_{k-p}}\delta(k_0-p_0-\omega_{k-p})2\pi\delta(k^2-m^2)\Bigg].\end{aligned} \tag{1.66}$$

Now, we can simplify this expression a bit more, realizing that the Lorentz-invariant delta functions can be split as follows

$$\begin{aligned}\delta(k^2-m^2) &= \delta((k_0-\omega_k)(k_0+\omega_k)) = \frac{1}{2\omega_k}\left[\delta(k_0-\omega_k)\theta(k_0)+\delta(k_0+\omega_k)\theta(-k_0)\right] \\ \delta((k-p)^2-m^2) &= \frac{1}{2\omega_{k-p}}\left[\delta(k_0-p_0-\omega_{k-p})\theta(k_0-p_0)+\delta(k_0-p_0+\omega_{k-p})\theta(p_0-k_0)\right]\end{aligned} \tag{1.67}$$

We can substitute these equations into eq. (1.66) to find four terms, which look somewhat similar to eq. (1.64):

$$\begin{aligned}\text{Disc}\left[i\mathcal{M}_s(s)\right] = -\lambda^2\Bigg[&\int \frac{d^Dk}{(2\pi)^D}\frac{\pi}{\omega_k}\delta(k_0-\omega_k)\frac{\pi}{\omega_{k-p}}\delta(k_0-p_0-\omega_{k-p})\theta(k_0-p_0) \\ &+ \int \frac{d^Dk}{(2\pi)^D}\frac{\pi}{\omega_k}\delta(k_0-\omega_k)\frac{\pi}{\omega_{k-p}}\delta(k_0-p_0+\omega_{k-p})\theta(p_0-k_0) \\ &+ \int \frac{d^Dk}{(2\pi)^D}\frac{\pi}{\omega_{k-p}}\delta(k_0-p_0-\omega_{k-p})\frac{\pi}{\omega_k}\delta(k_0-\omega_k)\theta(k_0) \\ &+ \int \frac{d^Dk}{(2\pi)^D}\frac{\pi}{\omega_{k-p}}\delta(k_0-p_0-\omega_{k-p})\frac{\pi}{\omega_k}\delta(k_0+\omega_k)\theta(-k_0)\Bigg].\end{aligned} \tag{1.68}$$

In the same way as we argued for eq. (1.64) to be zero, we can prove that three out of these four terms (i.e. all except the second) are also zero. In fact, going to the center of mass frame $p^\mu = (\sqrt{s}, \vec{0})$ and using $\omega_{k-p} = \sqrt{(\vec{k}-\vec{p})^2 + m^2} = \omega_k$ in this frame, we are left with, respectively

$$\int dk_0 \delta(k_0 - \omega_k)\delta(k_0 - p_0 - \omega_{k-p})\theta(k_0 - p_0) \to \int dk_0 \delta(k_0 - \omega_k)\delta(-\sqrt{s})\theta(k_0 - \sqrt{s}) = 0$$
$$\int dk_0 \delta(k_0 - p_0 - \omega_{k-p})\delta(k_0 - \omega_k)\theta(k_0) \to \int dk_0 \delta(-\sqrt{s})\delta(\omega_k - \omega_k)\theta(\omega_k) = 0$$
$$\int dk_0 \delta(k_0 - p_0 - \omega_{k-p})\delta(k_0 + \omega_k)\theta(-k_0) \to \int dk_0 \delta(-\sqrt{s} - 2\omega_k)\delta(k_0 + \omega_k)\theta(-\omega_k) = 0$$

all due to the lack of support for at least one delta functions in each integral, since both $\sqrt{s}$ and ω_k are always positive. Putting all together we are then left with

$$\text{Disc}\left[i\mathcal{M}_s(s)\right] = -\lambda^2 \left[\int \frac{d^D k}{(2\pi)^D} \frac{\pi}{\omega_k} \delta(k_0 - \omega_k) \frac{\pi}{\omega_{k-p}} \delta(p_0 - k_0 - \omega_{k-p}) \right]. \tag{1.69}$$

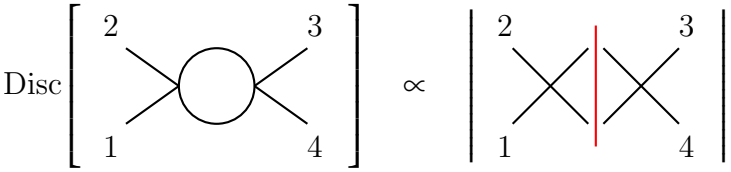
Finally, we can use again eq. (1.67) to render this expression Lorentz invariant by realizing that

$$\theta(k_0)\delta(k^2 - m^2) = \frac{1}{2\omega_k}\delta(k_0 - \omega_k)$$
$$\theta(p_0 - k_0)\delta((p-k)^2 - m^2) = \frac{1}{2\omega_{k-p}}\delta(p_0 - k_0 - \omega_k)$$

so that the discontinuity can be written as

$$\begin{aligned}\text{Disc}\left[i\mathcal{M}_s(s)\right] &= -\lambda^2 \int \frac{d^D k}{(2\pi)^D} \left[(2\pi)\delta(k^2 - m^2)\theta(k_0)\right]\left[(2\pi)\delta((p-k)^2 - m^2)\theta(p_0 - k_0)\right] \\ &= -\lambda^2 \int \frac{d^D k}{(2\pi)^D} \left[(2\pi)\delta^{(+)}(k^2 - m^2)\right]\left[(2\pi)\delta^{(+)}((p-k)^2 - m^2)\right]. \end{aligned} \tag{1.70}$$

The operation of substituting a propagator with a delta function goes under the name of a *unitarity cut*. One also says that the cut propagator is put "on shell". Usually it is represented graphically as a proper "cut" through the respective propagators, i.e.



(1.71)

We can summarize the result for the discontinuity of the graph $\mathcal{M}_s(s)$ in the variable s, as given in eq. (1.70), by the following *semi-graphical* "cutting rules", first described by Cutkosky:

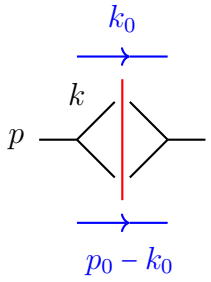
Cutkosky Rules:

1. To obtain the discontinuity of a graph in a kinematical variable (say s), "cut" a set of propagators through the diagram in any possible way, such that the diagram is split into two different parts, with the variable s flowing from one part to the other.

2. Replace every cut propagator using the following rule

$$\begin{aligned} \frac{i}{k^2 - m^2 + i\epsilon} &\to +(2\pi)\delta(k^2 - m^2)\theta(k_0) \\ \frac{1}{k^2 - m^2 + i\epsilon} &\to -(2\pi i)\delta(k^2 - m^2)\theta(k_0) \end{aligned} \tag{1.72}$$

   In doing this, *pay attention to the directionality of the cut: the energy flows from the left to the right of the cut:*



(1.73)

3. To obtain the discontinuity *(or the imaginary part)* in the variable s, sum over all allowed cuts.

Of course, these rules can be used to compute the *imaginary* part of a graph, possibly generated by multiple variables going "above threshold". For example, if we are dealing with a triangle graph, the latter depends in general on three different kinematical invariants, and one can compute the imaginary part in each of them by summing all relevant cuts. We haven't really demonstrated these rules in full generality, but we have hopefully provided a practical exemplification.[2]

1.3 Poles of Green Functions

Branch cuts are one type of singularities that can appear in Feynman diagrams and scattering amplitudes, but not the only one. As you must have seen explicitly when you computed your very first tree-level Feynman diagram in QFT, a single virtual particle exchange can

[2] If you are interested in a detailed proof, you could try to look into the *Largest-time equation*, see "Unitarity and causality in a renormalizable field theory with unstable particles" by Tini Veltman on *Physica 29 (1963) 186-207* and "Dispersion relations for Feynman Graphs", by E. Remiddi on *Helv.Phys.Acta 54 (1982) 364*.

produce a "pole" when the corresponding particle goes on-shell. For example

$$\lim_{q^2\to 0}\left[\begin{array}{ccc} p_1 & & p_3 \\ & \diagdown\quad\diagup & \\ & q & \\ & \diagup\quad\diagdown & \\ p_2 & & p_4 \end{array}\right] \sim \quad \frac{1}{q^2} \qquad \text{where } \; q = p_1 + p_2\,. \tag{1.74}$$

In fact, the relation between *poles* and intermediate *one-particle state going on shell* in arbitrary N-point correlation function (or Green functions) is a very general result. As the optical theorem discussed above, it is a rare non-perturbative result, and in fact it does not require the one-particle intermediate states to be "elementary": they could be bound states! For example, in specific situations, a positronium bound state can be exchanged as a single-particle intermediate state, leaving a trace through a pole at the positronium mass in the corresponding correlator.

The proof can be found on Weinberg, Volume 1 (Section 10.2) and on Schwartz (Section 24.3). The proof is tedious and we won't reproduce it here. Instead, we will summarize this important result and then see it "in action" in the special case of two-point correlators. This is something you should already be familiar with from your previous QFT studies and that goes under the name of the "spectral representation" for the two-point function.

Let us start with the momentum-space Green function for n fields $\phi(x_1)$,...,$\phi(x_n)$:

$$G_n(p_1,...,p_n) = \int d^4x_1 e^{ip_1x_1}... \int d^4x_n e^{ip_nx_n} \langle\Omega|\,\mathrm{T}\,\{\phi(x_1)...\phi(x_n)\}\,|\Omega\rangle \tag{1.75}$$

where $|\Omega\rangle$ is the vacuum of the theory and T is the usual time-ordering operator. We divide now the momenta in two different sets

$$\left\{\begin{array}{c} p_1 \\ ... \\ p_r \end{array}\right\}_{\text{incoming}}, \qquad \left\{\begin{array}{c} p_{r+1} \\ ... \\ p_n \end{array}\right\}_{\text{outgoing}}, \tag{1.76}$$

the first set incoming and the second outgoing, and define the intermediate momentum

$$p = p_1 + ... + p_r = p_{r+1} + ... + p_n\,.$$

We assume now that there exists a *one-particle state* in the Hilbert space (not necessarily an elementary one!) $|\Psi\rangle$, with mass m_Ψ, which can be produced from the vacuum $|\Omega\rangle$ by action of the field operators, i.e.

$$\langle\Psi|\phi(x_1)...\phi(x_r)|\Omega\rangle \neq 0\,. \tag{1.77}$$

In other words, $|\Psi\rangle$ has non-zero matrix element with the vacuum through the operators associated to the first r momenta. Now, if this is the case, one can prove that the Green function has a pole at $p^2 = m_\Psi^2$ and, when $p^2 \to m_\Psi^2$, it factorises as follows

$$\begin{aligned} G_n(p_1,...,p_n) \to\; & (2\pi)^4\delta^{(4)}(p_1 + ... + p_r - p_{r+1} - ... - p_n) \\ & \times \underbrace{\mathcal{M}_\Psi^{1,r}\frac{i}{p^2 - m_\Psi^2 + i\epsilon}\mathcal{M}_\Psi^{r+1,n}}_{\text{sum over all intermediate states (hel, color, etc)}} \qquad + \text{non divergent terms} \end{aligned} \tag{1.78}$$

where $\mathcal{M}_{\Psi}^{1,r}$ is the matrix element for $\phi_1 + ... + \phi_r \to \Psi$

$$(2\pi)^4\delta^{(4)}(p_1 + ... + p_r - p_\Psi)\mathcal{M}_{\Psi}^{1,r} = \int d^4x_1...d^4x_r e^{ip_1x_1}...e^{ip_rx_r} \times \langle\Omega|\,\mathrm{T}\,\{\phi(x_1)...\phi(x_r)\}\,|\Psi\rangle \tag{1.79}$$

and similarly for $\mathcal{M}_{\Psi}^{r+1,n}$ as matrix element for $\Psi \to \phi_{r+1} + ... + \phi_n$.

In conclusion, from the last two sections, we can draw the following important lessons:

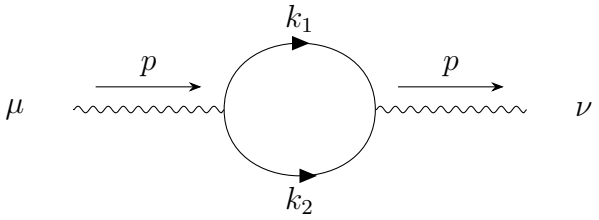
Scattering amplitudes are *analytic functions* of the kinematical invariants on which they depend, except for possible *poles* and *branch cuts*, that appear for very specific values of the invariants, determined by unitarity and locality:

- Poles can appear when single-particle states go on-shell, which already happens in tree-level graphs *(which can be seen also as a consequence of locality).*
- Branch cuts can appear when two or more internal propagators go on shell and so they first appear in loop diagrams *(as we have seen, a consequence of unitarity).*

In the following sections, we will discuss various important consequences of these findings, which will start to give us a glimpse of how "on-shell" physics can constrain "off-shell" physics.

1.4 Polarization sums and propagators

A first interesting consequence of unitarity, the you might have already encountered "in disguise", is about the explicit form of the numerator of fields' propagators. Indeed, you might have already observed that the numerator of the propagator for given field, is very similar (or better, identical!) to the expression for the *sum over physical polarizations* of the corresponding particle. Let's see this in the explicit example of the QED one-loop self energy graph



Modulo some numerical prefactors, the Feynman diagram above gives the following expression for the self-energy (without resolving the four-momentum conservation delta

function)

$$A = C\,(-1)\int \frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{\mathrm{Tr}\left[\gamma^\mu(-\not{k}_2+m)\gamma^\nu(\not{k}_1+m))\right]}{(k_1^2-m^2)(k_2^2-m^2)}(2\pi)^D\delta^{(D)}(k_1+k_2-p)$$
$$= C\int \frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\frac{\mathrm{Tr}\left[\gamma^\mu(\not{k}_2-m)\gamma^\nu(\not{k}_1+m))\right]}{(k_1^2-m^2)(k_2^2-m^2)}(2\pi)^D\delta^{(D)}(k_1+k_2-p) \tag{1.80}$$

where the propagator of momentum k_2 indicates the positron, and we assigned for convenience its momentum to go in the opposite direction (the momentum conservation delta function takes care of assigning its right value $k_2 = p - k_1$). We collected some irrelevant prefactors in the constant C (coupling constants etc).

Now let us cut this diagram using Cutkosky rules. Even if we are dealing with spinors here, the computation of the discontinuity proceeds in the same way, since all contributions to possible branch cuts come from the scalar part of the propagators. We can then write

$$\mathrm{Disc}\,[A] = C\int \frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\,\mathrm{Tr}\left[\gamma^\mu(\not{k}_2-m)\gamma^\nu(\not{k}_1+m))\right](2\pi)^D\delta^{(D)}(k_1-k_2-p)$$
$$\times(2\pi)\delta^{(+)}(k_1^2-m^2)(2\pi)\delta^{(+)}(k_2^2-m^2) \tag{1.81}$$

According to the optical theorem eq. (1.15), this should be proportional to the decay rate for a photon decaying to an electron-positron pair of momenta k_1 and k_2 respectively, i.e. $p \to k_1 + k_2$. Since we want to compare this to the calculation in eq. (1.80), it becomes clear why we assigned the fermion momenta as we did there. In this case we find:

$$\Gamma(\gamma\to e^+e^-) = \frac{C}{p^2}\sum_{s,s'}\int\frac{d^{D-1}k_1}{(2\pi)^{D-1}}\frac{1}{2\omega_{k_1}}\int\frac{d^{D-1}k_2}{(2\pi)^{D-1}}\frac{1}{2\omega_{k_2}}(2\pi)^D\delta^{(D)}(k_1+k_2-p)$$
$$\times\,\overline{v}_s(k_2)\gamma^\mu u_{s'}(k_1)\overline{u}_{s'}(k_1)\gamma^\nu v_s(k_2) \tag{1.82}$$

and using

$$\int\frac{d^{D-1}q}{(2\pi)^{D-1}}\frac{1}{2\omega_q} = \int\frac{d^Dq}{(2\pi)^D}(2\pi)\delta(q^2-m^2)\theta(q_0)\,,$$
$$\sum_{s'}u_{s'}(k_1)\overline{u}_{s'}(k_1) = \not{k}_1 + m\,,\quad \sum_s v_s(k_2)\overline{v}_s(k_2) = \not{k}_2 - m$$

we get as expected

$$\Gamma(\gamma\to e^+e^-) = \frac{C}{p^2}\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\,\mathrm{Tr}\left[\gamma^\mu(\not{k}_2-m)\gamma^\nu(\not{k}_1+m))\right](2\pi)^D\delta^{(D)}(k_1+k_2-p)$$
$$\times(2\pi)\delta^{(+)}(k_1^2-m^2)(2\pi)\delta^{(+)}(k_2^2-m^2)\,. \tag{1.83}$$

So finally we can once more verify the optical theorem

$$\mathrm{Disc}\,[A] \propto p^2\,\Gamma(\gamma\to e^+e^-) \tag{1.84}$$

with p^2 the "invariant mass" of the photons that decays. The important thing to realize here, is that while the two expressions are identical, their numerators have very different

origins! When computing the discontinuity, the numerator came from tracing over the fermion propagators in the fermion loop. On the other hand, when computing the decay rate, the *very same numerator* was produced when computing the sum over the electron and positron polarizations! So here, indirectly, we have "proved" that: *the numerator of the propagator of a fermion, has to be equal to the sum over its polarizations!* The same proof can be repeated for particles of any spin. For the particularly interesting case of spin-1 particles, we have to distringuish massless and massive case

- **Massive spin 1.** Consider a spin-1 massive particle of mass m (i.e. a Proca field), we have

$$\sum_{j=1}^{3} \epsilon_j^\mu \epsilon_j^{\nu *} = -g^{\mu\nu} + \frac{p^\mu p^\nu}{m^2} \tag{1.85}$$

  and indeed for its propagator we have

$$\Pi^{\mu\nu}(p) = i \frac{\sum_{j=1}^{3} \epsilon_j^\mu \epsilon_j^{\nu *}}{p^2 - m^2 + i\epsilon} = -i \frac{g^{\mu\nu} - \frac{p^\mu p^\nu}{m^2}}{p^2 - m^2 + i\epsilon} \tag{1.86}$$

- **Massless spin 1.** For a massless particle, the situation is slightly less obvious due to gauge invariance. In fact, the statement above remains true as long as we sum only over *physical polarizations*, which are also the ones that contribute to a cross-section or a decay rate! This can be achieved working in a so-called *physical (non-covariant) gauge*. A famous choice, which will we say more about when discussing Yang-Mills theories, is the class of Axial Light-Cone gauges, where we specify a light-cone momentum n^μ with $n^2 = 0$ and quantize with respect to it. In this gauge, we have that only 2 physical polarizations propagate and

$$\sum_{j=1}^{2} \epsilon_j^\mu \epsilon_j^{\nu *} = -g^{\mu\nu} + \frac{p^\mu n^\nu + n^\mu p^\nu}{p \cdot n} \tag{1.87}$$

  and indeed for its propagator we have

$$\Pi^{\mu\nu}(p) = i \frac{\sum_{j=1}^{2} \epsilon_j^\mu \epsilon_j^{\nu *}}{p^2 - m^2 + i\epsilon} = -i \frac{g^{\mu\nu} - \frac{p^\mu n^\nu + n^\mu p^\nu}{p \cdot n}}{p^2 - m^2 + i\epsilon} \,. \tag{1.88}$$

1.5 The spectral decomposition "reloaded"

To see how poles and branch cuts of correlators come together, we reconsider the spectral decomposition of the two-point correlator, which you should have already seen in QFT 1. We will briefly recap what you should already know, and connect it to the considerations we have done in this chapter. Also here, we focus on a scalar field theory, but adding spin is conceptually straightforward.

We start from the definition of a two-field correlator and, as usual add an identity operator in the form of a complete sum of states in the Hilbert space

$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle = \sum_X \int dX \langle \Omega | \phi(x) | X \rangle \langle X | \phi(y) | X \rangle \tag{1.89}$$

At this point, we add two more identity operators written in terms of the momentum operator $\hat{P}^\mu$ as $1 = e^{i\hat{P}\cdot x}e^{-i\hat{P}\cdot x}$ and write

$$\langle\Omega|\phi(x)|X\rangle = \langle\Omega|e^{i\hat{P}\cdot x}e^{-i\hat{P}\cdot x}\phi(x)e^{i\hat{P}\cdot x}e^{-i\hat{P}\cdot x}|X\rangle\,. \tag{1.90}$$

Using $\hat{P}^\mu|X\rangle = p_X^\mu|X\rangle$ and $\hat{P}^\mu|\Omega\rangle = 0$ (the vacuum has zero momentum), and that $\hat{P}$ generate translations, i.e.

$$e^{-i\hat{P}\cdot x}\phi(x)e^{i\hat{P}\cdot x} = \phi(0)\,,$$

we find

$$\langle\Omega|\phi(x)|X\rangle = e^{-ip_X\cdot x}\langle\Omega|\phi(0)|X\rangle \tag{1.91}$$

such that the two-point correlator becomes

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_X \int dX e^{-ip_X\cdot(x-y)}|\langle\Omega|\phi(0)|X\rangle|^2\,. \tag{1.92}$$

Now we can insert a delta-function $\delta^{(4)}(p-p_X)$ and write

$$\begin{aligned}\langle\Omega|\phi(x)\phi(y)|\Omega\rangle &= \int \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\underbrace{\left[\sum_X \int dX(2\pi)^4\delta^{(4)}(p-p_X)|\langle\Omega|\phi(0)|X\rangle|^2\right]}_{F(p^2)}\\ &= \int \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}F(p^2)\,.\end{aligned} \tag{1.93}$$

What can we say about $F(p^2)$? Some properties are obvious:

1. $F(p^2)$ is a Lorentz scalar, i.e. it can only depend on p^2, as the notation suggests.
2. It must be zero for $p^2 < 0$ and $p_0 < 0$, and positive otherwise $F(p^2) \geq 0$, because the states are physical states and they cannot have negative energy (or be tachionic).

A parametrization that respects these requirements is

$$F(p^2) = (2\pi)\,\theta(p^0)\rho(p^2)\,, \tag{1.94}$$

where $\rho(p^2)$ is called the *spectral density* and it should fulfil

$$\rho(p^2) \in \mathbb{R}^+\,, \qquad \rho(p^2) = 0 \ \text{ for } \ p^2 < 0\,. \tag{1.95}$$

Putting all together we write

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \int \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}(2\pi)\theta(p_0)\rho(p^2)\,. \tag{1.96}$$

Till now we haven't done much. Let us not try to relate this formula to the Feynman propagator. To do that, it is convenient to introduce the function

$$D(x,y,m^2) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{e^{-ip\cdot(x-y)}}{2\omega_p} = \int \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}(2\pi)\theta(p_0)\delta(p^2-m^2)\,, \tag{1.97}$$

with $\omega_p = \sqrt{\vec{p}^2 + m^2}$, which, by definition, is also the two-point function in a *free scalar theory*

$$\langle\Omega|\phi_0(x)\phi_0(y)|\Omega\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{e^{-ik\cdot(x-y)}}{2\omega_k} = D(x,y,m^2)\,. \tag{1.98}$$

Now we go back to eq. (1.96) and introduce a "redundant" delta function $\delta(p^2-q^2)$, such that we can write

$$\begin{aligned}\langle\Omega|\phi(x)\phi(y)|\Omega\rangle &= \int_0^\infty dq^2\delta(p^2-q^2)\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}(2\pi)\theta(p_0)\rho(q^2)\\ &= \int_0^\infty dq^2\rho(q^2)\int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}(2\pi)\theta(p_0)\delta(p^2-q^2)\\ &= \int_0^\infty dq^2\rho(q^2)D(x,y,q^2)\,.\end{aligned} \tag{1.99}$$

Comparing eq. (1.98) with eq. (1.99), we see that in a free theory the spectral density is

$$\rho_0(q^2) = \delta(q^2-m^2)\,, \tag{1.100}$$

but we should stress that eq. (1.99) is much more general, it is valid non-perturbatively!

But we still aren't where we wanted to get. To connect this with the propagator, we need time-order this expression and compute

$$\langle\Omega|\,\mathrm{T}\,\{\phi(x)\phi(y)\}\,|\Omega\rangle = \int_0^\infty dq^2\rho(q^2)\underbrace{\left[D(x,y,q^2)\theta(x_0-y_0)+D(y,x,q^2)\theta(y_0-x_0)\right]}_{\Delta(x,y,q^2)}\,. \tag{1.101}$$

This new function $\Delta(x,y,q^2)$ is nothing but the Feynman propagator of a free scalar field of "mass" q^2 in coordinate space:

$$\Delta(x,y,q^2) = \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-q^2+i\epsilon}e^{ip\cdot(x-y)} \tag{1.102}$$

which substituted in eq. (1.101) gives finally

$$\begin{aligned}\langle\Omega|\,\mathrm{T}\,\{\phi(x)\phi(y)\}\,|\Omega\rangle &= \int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-y)}\underbrace{i\int_0^\infty dq^2\frac{\rho(q^2)}{p^2-q^2+i\epsilon}}_{\Pi(p^2)}\\ &= \int\frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-y)}\,\Pi(p^2)\,.\end{aligned} \tag{1.103}$$

In this manipulations, we have found the *spectral* or *Källen-Lehmann representation* of the propagator

$$\Pi(p^2) = i\int_0^\infty dq^2\frac{\rho(q^2)}{p^2-q^2+i\epsilon} \tag{1.104}$$

which of course makes sense, as in the free theory using eq. (1.100), we have

$$\Pi_0(p^2) = \frac{i}{p^2 - m^2 + i\epsilon} \tag{1.105}$$

as expected. We also know that $\rho(p^2) > 0$ and real, so if we want to compute the discontinuity of this propagator we can do

$$\begin{aligned} \text{Disc}\left[\Pi(p^2)\right] &= \int_0^\infty dq^2 \rho(q^2) \text{Disc}\left[\frac{i}{p^2 - q^2 + i\epsilon}\right] = \int_0^\infty dq^2 \rho(q^2) 2\pi\delta(p^2 - q^2) \\ &= 2\pi\rho(p^2)\,, \\ \to \rho(p^2) &= \frac{1}{2\pi}\text{Disc}\left[\Pi(p^2)\right] \end{aligned} \tag{1.106}$$

where I used eq. (1.57). This means that the spectral density also provides the discontinuity (or, equivalently, the imaginary part) of the full propagator! Unitarity requires that the imaginary part of the propagator be related to the existence of intermediate particles that can go on-shell (or "be cut"), which in turn tells us that the spectral density contains information about all "particles" in the theory, whether they are elementary or not.

Let's be more specific. We go back to the definition of the spectral function

$$(2\pi)\rho(q^2)\theta(q_0) = \sum_X \int dX (2\pi)^4 \delta^{(4)}(q - p_X) |\langle\Omega|\phi(0)|X\rangle|^2 \tag{1.107}$$

and assume that in the spectrum of the theory there is one single-particle "scalar" state with mass m and momentum p_1. Let's say this corresponds to the first value of $X = 1$ and call the state $|1_{p_1}\rangle$. We separate this state from the sum and write

$$\sum_X \int dX = \int \frac{d^4p_1}{(2\pi)^4}(2\pi)\delta^{(+)}(p_1^2 - m^2) + \sum_{X\neq 1}\int dX \tag{1.108}$$

and we split similarly the spectral function

$$\rho(q^2)\theta(q_0) = \left[\rho_1(q^2) + \rho_{X\neq 1}(q^2)\right]\theta(q_0)\,. \tag{1.109}$$

By comparing this equation to eqs. (1.107) and (1.109) and canceling the (2π) from lhs and rhs, we can read off

$$\begin{aligned} \rho_1(q^2)\theta(q_0) &= \int \frac{d^4p_1}{(2\pi)^4}\delta^{(+)}(p_1^2 - m^2)|\langle\Omega|\phi(0)|1_{p_1}\rangle|^2(2\pi)^4\delta^{(4)}(q - p_1) \\ &= \delta(q^2 - m^2)\theta(q_0)|\langle\Omega|\phi(0)|1_q\rangle|^2\,. \end{aligned} \tag{1.110}$$

Finally, let us substitute this form of the spectral function in the Källen-Lehmann representation for the propagator eq. (1.104) to find

$$\Pi(p^2) = i\int_0^\infty dq^2 \frac{\rho(q^2)}{p^2 - q^2 + i\epsilon} = \underbrace{i\frac{|\langle\Omega|\phi(0)|1_p\rangle|^2}{p^2 - m^2 + i\epsilon}}_{\text{single-particle pole}} + \underbrace{i\int_0^\infty dq^2 \frac{\rho_{X\neq 1}(q^2)}{p^2 - q^2 + i\epsilon}}_{\text{contribution from multiparticle states}}\,. \tag{1.111}$$

This elucidates the "theorem" discussed in section 1.3 for the special case of two-point correlators. We have demonstrated, in fact, that for every single-particle state, a pole appears in the two-point correlator.

We also recall here that $|\langle\Omega|\phi(0)|1_p\rangle|^2 = Z$ is the wave-function renormalization constant (it tells us how the asymptotic state differs from the same state created from the vacuum)

$$\Pi(p^2) = \frac{iZ}{p^2 - m^2 + i\epsilon} + \text{multiparticle states} \tag{1.112}$$

where $Z = 1$ in the pole-mass (or on-shell) renormalization scheme, if m is also the on-shell mass. Indeed, we could have performed the same derivation also for the bare fields $\phi_b(x)$, and we would have found a very similar result, modulo the overall Z

$$\phi_b(x) = \sqrt{Z}\phi(x) \quad \rightarrow \quad \langle\Omega|\,\mathrm{T}\,\{\phi_b(x)\phi_b(y)\}\,|\Omega\rangle = Z\langle\Omega|\,\mathrm{T}\,\{\phi(x)\phi(y)\}\,|\Omega\rangle\,. \tag{1.113}$$

1.5.1 Spectral density in a scalar theory

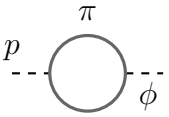
Let's see how this all discussion works in an explicit example. We consider a Lagrangian with two scalar fields, ϕ and π, with masses M and m respectively, with interaction Lagrangian

$$\mathcal{L}_{int} = \frac{\lambda}{2}\phi\pi^2\,.$$

The propagator of the field ϕ can then be written as always as

$$\Pi_\phi(p^2) = \frac{i}{p^2 - \overline{M}^2 + i\mathrm{Im}\left[\Sigma(p^2)\right]} \tag{1.114}$$

where $\overline{M}$ is the pole-mass renormalized according to the *real pole-mass scheme* defined in eq. (1.22). We can easily compute $\Sigma(p^2)$ to one-loop order, which corresponds to the Feynman diagram

$i\Sigma(p^2) =$ 

(1.115)

which consists in the very same calculation performed in section 1.2.1, see eq. (1.44), to obtain

$$\mathrm{Im}\left[\Sigma(p^2)\right] = \frac{\lambda^2}{32\pi}\sqrt{1 - \frac{4m^2}{p^2}}\theta(p^2 - 4m^2)\,. \tag{1.116}$$

Now using eq. (1.106), expanding order by order in λ, we can finally write

$$\begin{aligned}\rho(p^2) &= \frac{1}{2\pi}\mathrm{Disc}\left[\frac{i}{p^2 - \overline{M}^2 + i\mathrm{Im}\left[\Sigma(p^2)\right] + i\epsilon}\right] \\ &= \frac{1}{2\pi}\left(\mathrm{Disc}\left[\frac{i}{p^2 - \overline{M}^2 + i\epsilon}\right] - i\,\mathrm{Disc}\left[\frac{i\mathrm{Im}\left[\Sigma(p^2)\right]}{(p^2 - \overline{M}^2 + i\epsilon)^2}\right]\right) + \mathcal{O}(\lambda^4)\end{aligned} \tag{1.117}$$

where, even if it looks counterintuitive, we extracted the i from the Disc operation acting on the second term. The reason is that Disc is defined as the difference of the argument in the brackets, below and above the branch cut generated by the pole of the propagator. This requires only to change the sign of $i\epsilon$ and $i\mathrm{Im}\left[\Sigma(p^2)\right]$, and not of the overall i! Using then

$$\mathrm{Disc}\left[\frac{i}{p^2-\overline{M}^2+i\epsilon}\right]=2\pi\delta(p^2-\overline{M}^2)$$

and

$$\mathrm{Disc}\left[\frac{i\mathrm{Im}\left[\Sigma(p^2)\right]}{(p^2-\overline{M}^2+i\epsilon)^2}\right]=2i\frac{\mathrm{Im}\left[\Sigma(p^2)\right]}{(p^2-\overline{M}^2)^2}+\mathcal{O}(\epsilon^2) \tag{1.118}$$

we finally obtain

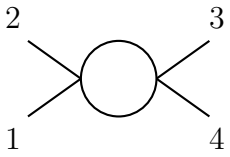
$$\rho(p^2)=\underbrace{\delta\left(p^2-\overline{M}^2\right)}_{\text{pole at one-particle state}}+\underbrace{\frac{\lambda^2}{32\pi^2}\sqrt{1-\frac{4m^2}{p^2}}\theta(p^2-4m^2)\frac{1}{\left(p^2-\overline{M}^2\right)^2}}_{\text{branch cut at the first multi-particle threshold}}+\mathcal{O}(\lambda^4)\,. \tag{1.119}$$

In eq. (1.119) we see how the spectral density, as expected, "knows" about the one-particle pole and the multi-particle branch cuts. Expanding to higher orders in perturbation theory, more multiparticle threshold can open, generating more branch cuts. We also notice that the residue at the pole is normalized to 1, as expected since we are working in the pole-mass scheme. The same would not be true in a different renormalization scheme!

1.6 Dispersion Relations

We have said a lot about discontinuities, how to compute them, and how they are related to internal virtual particles going on-shell. Of course, one important practical reason to be interested in discontinuities is that, typically, they are *simpler to compute* than full graphs, through the cutting rules provided above. Moreover, as we will see in this sections, they allow us to reconstruct the full "real" part of the graph through a so-called *dispersion relation*.[3] This is a first example of how "on-shell" quantities (cut diagrams) puts direct "constraints" on off-shell ones (uncut diagrams).

To see how this work, let us go back to our graph

$i\mathcal{M}_s(s)=$ 

. (1.120)

[3]The nomenclature "dispersion relations" is derived from optics and in particular the use of Kramers–Kronig relations in their context.

As a consequence of the considerations done in the previous sections, this function can develop a branch cut when $s \geq 4m^2$ but it has no poles (there is no one-particle state propagating from left to right). Apart from this, it is an *analytic function* of s. Let us consider than an arbitrary point s in the s-complex plane, importantly *not exactly on the branch cut*, see fig. **??**ADD FIGURE. As usual, if we are interested in a point on the branch cut, we must specify on which side of it we are. This is specified by Feynman's prescription and it turns out that there is an easy mnemonic rule to pick the right choice:

Every Mandelstam invariant that is above the threshold to produce an imaginary part, must carry a positive $i\epsilon$, so in our case for example $s = 5m^2 + i\epsilon$. This is equivalent to Feynman's prescription.

Clearly, since the function is analytic, we can write

$$i\mathcal{M}_s(s) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{ds'}{s'-s}(i\mathcal{M}_s(s')) \tag{1.121}$$

where the contour $\mathcal{C}$ encircles the point s, without crossing the branch-cut. We can now deform this contour to infinity, encircling the branch-cut, to find the equivalent representation

$$i\mathcal{M}_s(s) = \frac{1}{2\pi i}\left[\int_{\mathcal{C}_\infty} + \int_{\gamma_0}\right]\frac{ds'}{s'-s}(i\mathcal{M}_s(s')) + \frac{1}{2\pi i}\left[\int_{4m^2+i\delta} - \int_{4m^2-i\delta}\right]\frac{ds'}{s'-s}(i\mathcal{M}_s(s'))\,. \tag{1.122}$$

Now the contour γ_0 can be shrunk to zero and the contribution from $\int_{\gamma_0}$ disappears. The last term instead, can be easily seen to define precisely the discontinuity of the function across the branch cut

$$\begin{aligned}\left[\int_{4m^2+i\delta} - \int_{4m^2-i\delta}\right]\frac{ds'}{s'-s_0}(i\mathcal{M}_s(s')) &= \int_{4m^2}^{\infty}\frac{ds'}{s'-s_0}\left(i\mathcal{M}_s(s'+i\delta) - i\mathcal{M}_s(s'-i\delta)\right)\\ &= \int_{4m^2}^{\infty}\frac{ds'}{s'-s_0}\mathrm{Disc}\left(i\mathcal{M}_s(s)\right)\end{aligned} \tag{1.123}$$

and finally using $\mathrm{Disc} = 2i\,\mathrm{Im}$ we get our final formula

$$i\mathcal{M}_s(s) = \frac{1}{2\pi i}\underbrace{\int_{\mathcal{C}_\infty}\frac{ds'}{s'-s}(i\mathcal{M}_s(s'))}_{=0 \text{ if } \lim_{s\to\infty}\mathcal{M}_s(s)\sim 1/s} + \frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'-s}\underbrace{\mathrm{Im}\left[i\mathcal{M}_s(s')\right]}_{\text{from Cutkosky rules}}\,. \tag{1.124}$$

One can prove that if non-zero, the term at infinity

$$\int_{\mathcal{C}_\infty}\frac{ds'}{s'-s}(i\mathcal{M}_s(s')) = \mathcal{P}(s)$$

where $\mathcal{P}(s)$ is a polynomial in s. This term is related to the UV behavior of the graph, which in turn can be improved by a so-called "subtraction": instead of considering $i\mathcal{M}_s(s)$,

we compute $i\mathcal{M}_s(s) - i\mathcal{M}_s(s_0)$, for an arbitrary point s_0. Repeating the same steps as above, we obtain

$$i\mathcal{M}_s(s) - i\mathcal{M}_s(s_0) = \frac{1}{2\pi i}\int_{\mathcal{C}_\infty} ds'(i\mathcal{M}_s(s'))\frac{(s-s_0)}{(s'-s)(s'-s_0)} + \frac{1}{\pi}\int_{4m^2}^{\infty} ds' \mathrm{Im}\left[i\mathcal{M}_s(s')\right]\frac{(s-s_0)}{(s'-s)(s'-s_0)}\,, \tag{1.125}$$

where both terms have now improved convergences as $s \to \infty$, and therefore it is enough for $\mathcal{M}_s(s) \sim s^0$ as $s \to \infty$, for the dispersion relation to converge and to be able to prove that

$$\int_{\mathcal{C}_\infty} ds'(i\mathcal{M}_s(s'))\frac{(s-s_0)}{(s'-s)(s'-s_0)} \to 0\,.$$

Subtracting the graph at a specific value of $s = s_0$ reminds of the important issue of UV renormalization: we often define renormalized quantities (for example in the on-shell or pole scheme) by providing their value at a given kinematical point, for example $p^2 = s = 0$. Dispersion relations, in this sense, allow us to naturally relate the UV behavior of the graph with the counter-term required to render it finite! We will see an explicit example of this in the exercises.

Chapter 2

On-shell methods

In the previous lectures, we have discussed the consequences of Unitarity, Locality and Causality on Green Functions. We have re-discovered the Optical Theorem, which relates imaginary parts of forward amplitudes to cross-sections, and we derived Cutkosky rules, which relate discontinuities to "cuts", i.e. virtual particles going on-shell. Finally, we presented a general argument about poles of correlators, relating their appearance to the existence of (on-shell) single-particle states in the spectrum of the theory.

The word "on-shell" appeared a lot in the previous paragraph, and indeed this is just a small fraction of the large amount of evidence accumulating towards the idea that *very general properties of QFT are entirely determined by on-shell physics*. This logic is somewhat opposite to that of fields and Lagrangians, on which standard QFT is built. In fact, in the standard approach, everything is "off-shell" (we build Green functions from off-shell Feynman diagrams), and we only go "on-shell" at the very end of the calculation, when we "amputate" the Green functions and use the external polarization vectors (spinors etc) to represent the on-shell external states.

The idea behind "on-shell methods", is to see how far we can go in deriving predictions from QFT, without ever having to talk about Lagrangians, fields and off-shell correlators. These methods are very promising, but they are far from being mature enough to substitute a fields-based approach to QFT. Nevertheless, some of their predictions go beyond what we can even dream of achieving in the standard off-shell approach, and for this reason, I think they deserve to be studied already in a "standard course" like this one. This is literally a booming field, and we won't be able to go deep into everything that has been discovered in the past three decades, as many topics are still matter of research while these notes are being written. If these topics sound fascinating to you, you'll be able to deepen your understanding of them in more advanced specialized classes on the theory of scattering amplitudes. Our goal here, is to present a powerful formalism that allows us to make calculations of directly on-shell quantities (as scattering amplitudes) much simpler. This the by now famous *Spinor-Helicity formalism*. Armed with it, we will manage to derive the main result of this chapter: **the uniqueness of Yang-Mills theories**. What we mean with uniqueness will become more clear in the coming sections.

The material in this chapter is taken from various sources, including my own lecture

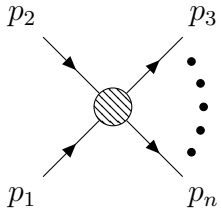
notes of my course of scattering amplitudes[1], lecture notes by Lance Dixon[2] and Kirill Melnikov, and the book *Quantum Field Theory and the Standard Model* by M. Schwartz.

2.1 Spinor Helicity Formalism

We start with a brief introduction to the formalism. Many of the properties will be worked out more in details in the exercises.

Let us start considering a generic S-matrix for the scattering of n particles of spins σ_i. For definiteness, let us say $p_1 + p_2 \to p_3 + ... + p_n$, such that

$$S = \langle \Psi^{out}_{p_3\sigma_3 \ldots p_n\sigma_n} | \Psi^{in}_{p_1\sigma_1 p_2\sigma_2} \rangle \quad = \tag{2.1}$$



where $|\Psi^{out}_{p_i\sigma_i \ldots p_j\sigma_j}\rangle$ are multi-particle states.

As you learned in QFT1, *one-particle* states are irreducible representations of the Poincaré group, i.e. under a Poincaré transformation

$$|\Psi_i\rangle \to P_{ij}|\Psi_j\rangle$$

where the irreps P_{ij} are, in turn, classified by the *irreducible representations* of the corresponding *Little Group* associated to each particle.

Briefly, the Little Group is defined as the subgroup of the Poincaré group that leaves the momentum of the particle, say p^μ, invariant. We have few possibilities, of which two are of interest for the elementary particles that we observe:

- if $p^\mu = 0$, there is only the vacuum and all is trivial and nothing is interesting.[3]
- if $p^\mu \neq 0$ and $p^2 > 0$, the particle is *massive*. We can go to its center-of-mass frame where $p^\mu = (m, \vec{0})$, from which it is clear that the subgroup that leaves this momentum invariant is the group of rotations in the three-dimensional space, $SO(3)$. Its irreps are, as we know, the same of $SU(2)$ and are labeled by an half-integer j, which we refer to as the "spin" of the particle. For each value of j, there are $(2j+1)$ different states, or "polarizations". For example, a massive particle of spin 1 has possible polarizations states "$\{-1, 0, 1\}$".
- if $p^\mu \neq 0$ and $p^2 = 0$, the particle is *massless*. We can orient its three-momentum along the z-axis and write $p^\mu = (E, 0, 0, E)$. Despite appearances, from here one can derive that the full subgroup of transformations that leaves this momentum invariant is the

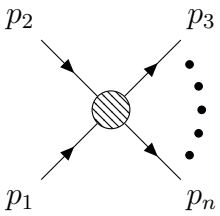
[1] See link.
[2] See, for example, link
[3] At least as far as this argument goes. The vacuum of a QFT can hide a lot of surprises...

so-called Euclidean group $ISO(2)$, i.e. the group of translations and rotations on the plane. Translations would generate a continuous eigenvalue to label the particles, which we don't observe. Focusing on $SO(2)$, its irreps are the same as $U(1)$ and are labeled in general by a continuous eigenvalue h, such that a general transformation can be parametrized as $g = e^{ih\theta}$ in terms of an angle θ. This is again a continuous eigenvalue that we don't seem to observe. In fact, a standard argument on the non-trivial topology of the Lorentz group, allows us to argue that only those h are allowed, for which a 4π rotation would take us back to the starting point

$$e^{ih(4\pi)} = 1\,,$$

which imposes $h = \frac{n}{2}$ with $n \in \mathbb{Z}$, i.e. also in this case, h has to be an half-integer. So massless particles should exists labeled by a single eigenvalue, which takes one specific value (integer or half-integer), which we call *helicity*. The fact that particles like the photon or massless fermions appear with two helicities (the photon can have ± 1), comes from the extra requirement of parity invariance: if we want a parity invariant theory (as QED), we need the particle to exist in both helicities, since a parity transformation sends one to the other.

As you should have seen in QFT 1, the transformation properties of single- and multi-particle states, implies an important consequence for the S-matrix eq. (2.1): while we are used to think about it as a Lorentz scalar, it is actually not invariant under Poincaré transformation, instead it is "covariant" under Little Group transformations. One could say that through its transformation properties, the S-matrix "knows" the "spin" of the incoming and outgoing particles. In order words, think about some generic amplitude as in eq. (2.1), where, for definiteness, all external particles are spin 1. In the standard Feynman-diagrams based approach, the amplitude will take the general form



$$= \underbrace{\epsilon_1^{\mu_1 *}\epsilon_2^{\mu_2 *}\epsilon_3^{\mu_3}\ldots\epsilon_n^{\mu_n}}_{\text{polarization vectors}}\ \underbrace{\mathcal{A}_{\mu_1\ldots\mu_n}}_{\text{Feyn. diags.}} \tag{2.2}$$

where $\mathcal{A}_{\mu_1\ldots\mu_n}$ is a Lorentz tensor, as the indices make manifest, and is computed through the LSZ formula, from off-shell (amputated) Feynman diagrams. The polarization vectors $\epsilon_j^{\mu_j}$, instead, have double transformation properties: the index μ_j says that they transform as Lorentz vectors, and in fact by contracting them with the tensor $\mathcal{A}_{\mu_1\ldots\mu_n}$, we obtain something that is scalar under Lorentz transformations. At the same time, they also must carry information from the Little Group of the corresponding particle, in order to guarantee that the final result for the S-matrix transforms as described above.

The Spinor-Helicity formalism that we aim to introduce in this section, has been developed to make these transformation properties manifest for *massless particles*. We will introduce the formalism having in mind QED with massless electrons as a physical theory which contains both spin 1/2 and spin 1 particles.

2.1.1 Spinor-Helicity for Weyl Spinors

We start considering the building blocks used to build massless spinors in QFT, namely the Weyl spinors. There are two types of Weyl spinors, which correspond to two distinct irreducible representations of the Lorentz group. We briefly recall here that the Lie Algebra of $SL(2,\mathbb{C})$, the covering group of the Lorentz group $SO(1,3)$, is isomorphic (or equivalent) to the Lie Algebra of $SU(2)\times SU(2)$. For this reason, each irreducible representation of the Lorentz group is labeled by two half integers (j_1, j_2). With this notation, the two irreducible representation we are interested in correspond to

- $(1/2, 0)$, called the Left-handed Weyl spinors.
- $(0, 1/2)$, called the Right-handed Weyl spinors.

Both spinors transform in the same way under spatial rotations

$$\psi_{L,R} \underbrace{\longrightarrow}_{\text{rotations}} \psi'_{L,R} = e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}\psi_{L,R}\,, \tag{2.3}$$

where σ_i are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.4}$$

On the contrary, Left- and Right-handed spinors transform in the opposite way under Lorentz boosts

$$\psi_{L,R} \underbrace{\longrightarrow}_{\text{boost}} \psi'_{L,R} = e^{\pm\vec{\phi}\cdot\frac{\vec{\sigma}}{2}}\psi_{L,R}\,. \tag{2.5}$$

Moreover, one can easily prove that

$$\begin{aligned} \overline{\psi}_L &= +i\sigma_2\psi_L^* \quad \text{transforms as a right-handed spinor} \\ \overline{\psi}_R &= -i\sigma_2\psi_R^* \quad \text{transforms as a left-handed spinor}\,, \end{aligned} \tag{2.6}$$

which means that by taking a complex conjugation and multiplying by the matrix $\pm i\sigma_2$, we can transform a Left-handed to a Right-handed spinor and viceversa. Notice the different sign in the second equation in eq. (2.6), which is not strictly needed for the transformation properties, but it is required to guarantee that if we concatenate the two transformations, we go back to the spinor we started from (Left → Right → Left), i.e.

$$\overline{\overline{\psi}_L} = -i\sigma_2\left[i\sigma_2\psi_L^*\right]^* = -i\sigma_2\left(i\sigma_2\right)\psi_L = \psi_L\,, \tag{2.7}$$

where we used $(i\sigma_2)^* = i\sigma_2$, since $i\sigma_2$ is real, and $\sigma_2\sigma_2 = 1$.

Since each spinor is a two-dimensional vector and in order to keep track of these different transformation properties, it is common to introduce two different sets of indices to indicate their components, referred to as "dotted" $(\dot{a})$ and "undotted" (a) indices. In particular we put (conventionally!)

$$\psi_L = \psi_a\,, \qquad \psi_R = \psi^{\dot{a}} \tag{2.8}$$

i.e. we assign lower undotted indices to Left-handed spinors, and upper dotted indices to Right-handed spinors. To express the transformation properties from Left to Right-handed indices eq. (2.6), we write

$$i\sigma_2(\psi_L)^* \longrightarrow (i\sigma_2)^{\dot{a}\dot{b}}(\psi_b)^* = (i\sigma_2)^{\dot{a}\dot{b}}\psi_{\dot{b}} = \psi^{\dot{a}} \longrightarrow \psi_R\,, \tag{2.9}$$

i.e. *complex conjugation* swaps dotted and undotted indices, which $(i\sigma_2)$ can be used to raise dotted indices, which suggests that it should take the role of a metric in the space of dotted indices.

Similarly, we write

$$-i\sigma_2(\psi_L)^* \longrightarrow (-i\sigma_2)_{ab}(\psi^{\dot{b}})^* = (-i\sigma_2)_{ab}\psi^b = \psi_a \longrightarrow \psi_L\,, \tag{2.10}$$

such that $(-i\sigma_2)$ is the metric in the space of undotted indices and can be used to lower undotted indices. Again, this suggests that this matrix coult take the role of the metric in the space of undotted indices. But how do we lower dotted indices and raise undotted ones, respectively? It's quite easy! Indeed, we $\pm i\sigma_2$ is real, and we have seen that complex conjugation must by consistency change dotted to undotted indices (and viceversa), we must have that

$$(i\sigma_2)^{\dot{a}\dot{b}} = \left((i\sigma_2)^{\dot{a}\dot{b}}\right)^* = (i\sigma_2)^{ab}\,, \qquad (-i\sigma_2)_{ab} = \left((-i\sigma_2)_{ab}\right)^* = (i\sigma_2)_{\dot{a}\dot{b}}\,. \tag{2.11}$$

This completes the definition of the *spinor metric*. It is also easy to see that its definition coincides with the Levi-Civita tensor in 2 dimensions, and so we also put

$$\begin{aligned}(i\sigma_2)^{\dot{a}\dot{b}} = (i\sigma_2)^{ab} = \varepsilon^{ab} = \varepsilon^{\dot{a}\dot{b}} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,,\\ (-i\sigma_2)_{\dot{a}\dot{b}} = (-i\sigma_2)_{ab} = \varepsilon_{ab} = \varepsilon_{\dot{a}\dot{b}} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,,\end{aligned} \tag{2.12}$$

such that

$$\begin{aligned}\varepsilon^{ab}\varepsilon_{bc} = (i\sigma_2)^{ab}(-i\sigma_2)_{bc} &= \delta^a_c\,,\\ \varepsilon^{\dot{a}\dot{b}}\varepsilon_{\dot{b}\dot{c}} = (i\sigma_2)^{\dot{a}\dot{b}}(-i\sigma_2)_{\dot{b}\dot{c}} &= \delta^{\dot{a}}_{\dot{c}}\,.\end{aligned} \tag{2.13}$$

Now, the use of dotted and undotted indices is based on conventions that can change from one book or article to the another, and for this reason they are often an infinite source of headaches and confusion. The good thing is that, for what we are interested in, with the spinor-helicity formalism we can completely forget about them!

We start by using the spinor metric eq. (2.12) to define a *scalar product* in spinor space. We consider here for spinors the spinor *wave functions*. These are the c-numbers that appear in the Fourier decomposition of spinor field and therefore, to avoid confusion, are *not Grassmann numbers*, but instead they normally commute!! We start with left-handed spinors and define

$$\psi^a\chi_a = \psi_a\chi_b\varepsilon^{ab}\,. \tag{2.14}$$

Our definition is clearly Lorentz invariant. One can also see that, while the spinors are c-functions, the spinor product is *antisymmetric*

$$\psi^a \chi_a = \psi_a \chi_b \varepsilon^{ab} = -\psi_a \chi_b \varepsilon^{ba} = -\chi^a \psi_a \,, \tag{2.15}$$

which means that in the definition of the spinor product, the order matters! This of course implies as well that

$$\psi^a \psi_a = 0 \,. \tag{2.16}$$

The same equations are true for dotted (right-handed) indices. There, we define a spinor product as

$$\psi_{\dot{a}} \chi^{\dot{a}} = \psi^{\dot{a}} \chi^{\dot{b}} \varepsilon_{\dot{a}\dot{b}} \,. \tag{2.17}$$

Again, the spinor product is antisymmetric and the order is therefore important.

2.1.2 Spinor-Helicity for Dirac spinors

We want now to use our Weyl spinors to build Dirac spinors, which are the basic building blocks for parity invariant theories (in which both helicities have to be paired to obtain a field which can is transformed to itself under parity transformations). Dirac spinors live in the direct sum representation $(0, 1/2) \oplus (1/2, 0)$.

We work in the chiral representation and define the usual left and right projectors

$$P_L = \frac{1 - \gamma_5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \qquad P_R = \frac{1 + \gamma_5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.18}$$

and define a Dirac spinor as

$$u(p) = N \begin{pmatrix} \psi_L(p) \\ \psi_R(p) \end{pmatrix} \tag{2.19}$$

where N is some normalization, whose exact value we will largely ignore in what follows. For definiteness, we assume that *all momenta are incoming* and define left and right-handed Dirac spinors as

$$u_L(p) = N \begin{pmatrix} \psi_L(p) \\ 0 \end{pmatrix} = N \begin{pmatrix} \psi_a \\ 0 \end{pmatrix} , \qquad u_R(p) = N \begin{pmatrix} 0 \\ \psi_R(p) \end{pmatrix} = N \begin{pmatrix} 0 \\ \psi^{\dot{a}} \end{pmatrix} \tag{2.20}$$

and similarly for the adjoint spinors

$$\overline{u}_L(p) = u_L^\dagger(p) \gamma_0 = N^* \left(0, \psi_{\dot{a}}\right) , \qquad \overline{u}_R(p) = u_L^\dagger(p) \gamma_0 = N^* \left(\psi^a, 0\right) . \tag{2.21}$$

Finally, in oder to completely forget about dotted and undotted indices, we introduce a bra-(c)-ket notation

$$u_L(p) = p] \,, \qquad \overline{u}_L(p) = \langle p \,, \qquad u_R(p) = p\rangle \,, \qquad \overline{u}_R(p) = [p \,. \tag{2.22}$$

These are typically referred to as *angle* and *square* brackets, in a obvious manner. I stress here that, depending on how formulas read better, I might or might now add a bar of the left or right of the spinor brackets, i.e. we will use interchangeably

$$k] = |k]\,, \quad [k = [k|\,, \qquad |k\rangle = k\rangle\,, \quad \langle k = \langle k|\,.$$

Using eq. (2.20), we can then immediately see that

$$\overline{u}_L(p)u_L(q) = \langle pq] = 0\,, \qquad \overline{u}_R(p)u_R(q) = [pq\rangle = 0\,, \qquad \forall\ p,q \tag{2.23}$$

while

$$\begin{aligned}\overline{u}_L(p)u_R(q) &= |N|^2\psi^a(p)\psi_a(q) = \langle pq\rangle \neq 0\,,\\ \overline{u}_R(p)u_L(q) &= |N|^2\chi_{\dot{a}}(p)\chi^{\dot{a}}(q) = [pq] \neq 0\,,\end{aligned} \tag{2.24}$$

where the spinors in each spinor product are different because they are evaluated at two different momenta! In this notation, antisymmetry gives

$$\langle pq\rangle = -\langle qp\rangle\,, \qquad [pq] = -[qp]\,, \qquad \langle pp\rangle = [pp] = 0\,, \qquad \forall\ \ p,q\,. \tag{2.25}$$

Here it is important to stress that some references use the "same" bracket notation for Weyl spinors instead of Dirac spinors, i.e. intending something along the lines of

$$u_L(p) \propto \begin{pmatrix} p] \\ 0 \end{pmatrix} \qquad \overline{u}_R(q) \propto \begin{pmatrix} [q\,, & 0 \end{pmatrix}\,.$$

Ultimately, the value of the spinor products is clearly the same, as one can see comparing with eqs. (2.20) and (2.21). The bracket notation, in which we carry only information on the momentum, is a very compact and convenient one and it's the one that one usually refers to as *spinor-helicity* formalism. Clearly, this can work in this form *only for massless spinors*. Generalizations for massive ones exists, but we will not consider them here.

We have seen which spinor products are zero eq. (2.25). What can say about the value of the ones that are non-zero?

- First, notice that if the momenta are real, then obviously

$$\langle pq\rangle^* = (\overline{u}_L(p)u_R(q)) = \overline{u}_R(q)u_L(p) = [qp] \tag{2.26}$$

  so, as expected from our previous construction, complex conjugation swaps angle and square brackets, and it also swaps the order of the spinors.

- The completeness relation for massless spinors reads in this notation

$$\sum_\lambda u_\lambda(q)\overline{u}_\lambda(q) = \not{q} = |q\rangle[q| + |q]\langle q|\,. \tag{2.27}$$

- Eq. (2.26) and eq. (2.27) together imply that

$$\langle pq\rangle[qp] = |\langle pq\rangle|^2 = |[pq]|^2\,, \tag{2.28}$$

but also

$$\langle pq\rangle[qp] = \langle p\big(|q\rangle[q|\big)p] = \langle p\big(|q\rangle[q| + \underbrace{|q]\langle q|}_{\text{adds zero due to eq. (2.23)}}\big)p] = \langle p\not{q}p]\,. \tag{2.29}$$

- Now notice that any "spinor string" that starts and ends with the same momentum can be turned into a trace

$$\langle p\not{q}p] = \underbrace{\sum_\lambda \overline{u}_\lambda(p)\left(\frac{1+\gamma_5}{2}\right)\not{q}\,u_\lambda(p)}_{\text{inserting } P_L \text{ allows me to sum over all helicities}} = \mathrm{Tr}\left[\not{p}\not{q}P_L\right] = 2\,p\cdot q\,. \tag{2.30}$$

- Finally, putting all the previous steps together, we see that we must have

$$\langle pq\rangle = \sqrt{|2p\cdot q|}e^{i\phi_{pq}}\,, \qquad [pq] = -\sqrt{|2p\cdot q|}e^{-i\phi_{pq}}\,, \tag{2.31}$$

which shows that spinor products are proportional to the square root of the scalar product of their momenta. Moreover, they are always defined modulo a phase, which is often referred to as "spinor phase". As we will see later, this phase is also connected to the *little group covariance* of the polarization vectors and spinors which we hinted to at the beginning.

Properties of spinor products

Let us list here some of the most important properties satisfied by spinor products. We will not prove them here, but instead they will be the subject of the second exercise sheet.

1. *Dirac Equation* $\not{p}u(p)$ reads in this formalism

$$\not{p}|p\rangle = \not{p}|p] = \langle p|\not{p} = [p|\not{p} = 0\,. \tag{2.32}$$

2. The left- and right-handed *projectors* read

$$|p\rangle[p| = \frac{1+\gamma_5}{2}\not{p}\,, \qquad |p]\langle p| = \frac{1-\gamma_5}{2}\not{p}\,. \tag{2.33}$$

3. Spinor products fulfill the so-called *Gordon identity*

$$\langle p\gamma^\mu p] = [p\gamma^\mu p\rangle = 2p^\mu\,. \tag{2.34}$$

4. They can be *conjugated* as follows

$$[p\gamma^\mu q\rangle = \langle q\gamma^\mu p]\,. \tag{2.35}$$

5. They fulfill *Fierz identities*

$$\begin{aligned}[p\gamma^\mu q\rangle[k\gamma_\mu l\rangle &= 2[pk]\langle lq\rangle \\ \langle p\gamma^\mu q][k\gamma_\mu l] &= 2\langle pk\rangle[lq]\,.\end{aligned} \tag{2.36}$$

6. They fulfill *Schouten identities*

$$\begin{aligned}\langle pq\rangle\langle kl\rangle &= \langle pk\rangle\langle ql\rangle + \langle pl\rangle\langle kq\rangle \\ [pq][kl] &= [pk][ql] + [pl][kq]\,.\end{aligned} \tag{2.37}$$

7. From the definition of the left- and right-handed projection operators, it should be obvious that

$$\begin{aligned}&\langle p\gamma^{\mu_1}...\gamma^{\mu_{2n}}q] = [p\gamma^{\mu_1}...\gamma^{\mu_{2n}}q\rangle = 0 && \text{even number of Dirac matrices} \\ &\langle p\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}q\rangle = [p\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}q] = 0 && \text{odd number of Dirac matrices}\end{aligned} \tag{2.38}$$

   i.e., spinor products which start and end with the same type of bracket, are zero if they contain an odd number of Dirac matrices. Similarly, the ones which start and end with different types of brackets, are zero if they contain an even number of Dirac matrices.

8. Finally, for the opposite cases to the ones above, i.e. the non-zero spinor products, we have *reversal identities*. For strings of *odd* numbers of Dirac matrices

$$\langle p\gamma^{\mu_1}...\gamma^{\mu_{2n+1}}q] = [q\,\gamma^{\mu_{2n+1}}...\gamma^{\mu_1}p\rangle\,, \tag{2.39}$$

   while for strings of *even* numbers of Dirac matrices

$$\begin{aligned}\langle p\gamma^{\mu_1}...\gamma^{\mu_{2n}}q\rangle &= -\langle q\,\gamma^{\mu_{2n}}...\gamma^{\mu_1}p\rangle\,, \\ [p\gamma^{\mu_1}...\gamma^{\mu_{2n}}q] &= -[q\,\gamma^{\mu_{2n}}...\gamma^{\mu_1}p]\,.\end{aligned} \tag{2.40}$$

Here we will prove only one, which is a good exemplification of how many proofs with spinor products can be attempted. We consider Schouten identities, i.e. point 6, and focus on the version with square brackets (of course the very same can be repeated for angle brackets). We want to prove

$$[pq][kl] = [pk][ql] + [pl][kq]\,.$$

This identity involves 4 different spinors. Its validity is based on the fact that spinors are actually *2-dimensional* objects and therefore, if the momenta are non-exceptional (not in some special parallel configuration), only two spinors can be independent in two dimensions. Let us pick as independent ones $|k]$ and $|l]$, so that we can write for a third spinor

$$|q] = A|k] + B|l]\,. \tag{2.41}$$

Contracting this with $[k$ and $[l$, respectively, and using anti-symmetry, we get two equations

$$[kq] = B[kl]\,, \qquad [lq] = A[lk] = -A[kl]\,, \tag{2.42}$$

which in turn implies

$$\begin{aligned} A &= -\frac{[lq]}{[kl]}\,, \qquad B = \frac{[kq]}{[kl]} \\ \to |q] &= -\frac{[lq]}{[kl]}|k] + \frac{[kq]}{[kl]}|l]\,. \end{aligned} \tag{2.43}$$

Now we multiply by $[p$ to finally obtain

$$\begin{aligned} [pq] &= -\frac{[lq]}{[kl]}[pk] + \frac{[kq]}{[kl]}[pl]\,. \\ \to [pq][kl] &= -[lq][pk] + [kq][pl] = [pk][ql] + [pl][kq]\,. \end{aligned} \tag{2.44}$$

2.1.3 Little-group scaling

As already anticipated, spinor products, among the other things, make little-group transformation properties manifest. We will argue now, in particular, that under a little-group transformation, the spinor brackets scale by a phase. This is consistent with the fact that spinor products per se are defined modulo a phase, as we saw in eq. (2.31). We start from the definition of the momentum in spinor space as

$$p^\mu \gamma_\mu = \not{p} = |p\rangle[p| + |p]\langle p| \tag{2.45}$$

From here, it's clear that a Lorentz transformation that leaves p^μ invariant can at most rescale the spinors by some complex number z. In particular, conventionally we choose

$$\begin{aligned} |p\rangle &\to z|p\rangle\,, \qquad [p| \to \frac{1}{z}[p|\,, \\ |p] &\to \frac{1}{z}|p]\,, \qquad \langle p| \to z\langle p|\,. \end{aligned} \tag{2.46}$$

That $|p\rangle$ and $\langle p|$ have to be shifted in the very same way should make sense if you think about their definition:

$$|p\rangle = u_R(p)\,, \qquad \langle p| = \overline{u}_L(p) = \overline{(-i\sigma_2)u_R^*} = [(-i\sigma_2)u_R^*]^\dagger\,\gamma_0 = \underbrace{u_R(p)}_{\text{all dep. on } p \text{ here}}\,(-i\sigma_2)^\dagger\gamma^0\,, \tag{2.47}$$

and similarly for $p]$ and $[p$.

Now, assuming the momentum p^μ is real , it's clear that

$$|p] = |p\rangle^* \;\longrightarrow\; z^*|p] \qquad \langle p| = [p|^* \;\longrightarrow\; \frac{1}{z^*}\langle p|\,. \tag{2.48}$$

Comparing these equations with eq. (2.46), we see immediately that for consistency they imply

$$z^* = \frac{1}{z} \;\longrightarrow\; |z|^2 = 1\,,$$

i.e. a Little-Group transformation must change the spinors by a phase!

2.1.4 Spinor-Helicity representation for spin-1 massless bosons

Before being able to use the formalism to do computation in realistic theories (and use it to make our general statements about Yang-Mills theories), we need to find a way to represent the other types of particles that we need in our theories: spin-1 massless bosons (like photons in QED and gluons in QCD, which we will encounter soon). The question is, how do we represent the polarization vector of a massless spin-1 boson? We will assume here to work with the spin-1 bosons of some gauge theory and, for definiteness, we will think of spinor QED, based on the gauge group $U(1)$. We will see later how these techniques will allow us to make very general statements for any gauge theory of a spin-1 massless boson.

Consider a photon of momentum p^μ. As you have learned in QFT 1, its polarization vector $\varepsilon^\mu_\lambda(p)$, with $\lambda = \pm$ fulfills the following properties

1. Transversality:
$$p_\mu \varepsilon^\mu_\lambda = 0 \tag{2.49}$$

2. Normalization:
$$\varepsilon^*_{\lambda\mu}\varepsilon^\mu_\lambda = -1 \tag{2.50}$$

Let us start with discussing an explicit representation for its polarization vectors, which you should have already encountered. As before, we assume all momenta are incoming and we rotate our reference frame such that $p^\mu = (E, 0, 0, E)$ (which we can do, because the boson is massless). There are two choices for the two physical polarizations of the vector boson, $\pm$, which in this frame are typically fixed as[4]

$$\varepsilon^\mu_+ = \frac{1}{\sqrt{2}}(0, 1, i, 0)\,, \qquad \varepsilon^\mu_- = \frac{1}{\sqrt{2}}(0, 1, -i, 0)\,. \tag{2.51}$$

From eq. (2.51) one can also explicitly see that, if one does not take the complex conjugation,
$$\varepsilon_{\mu\pm}\varepsilon^\mu_\pm = 0\,.$$

To consistently parametrize only the two physical polarizations, we need to specify an extra momentum r^μ. In fact, by making sure that $r\cdot p \neq 0$, p^μ and r^μ allow us to fully specify the 2-dimensional plane where the two physical polarizations "live". Following standard conventions, we do this imposing that

$$p^\mu \varepsilon^\pm_\mu = 0\,, \qquad r^\mu \varepsilon^\pm_\mu = 0\,, \qquad \text{with } \; r^2 = 0\,. \tag{2.52}$$

This choice determines what is usually called a (light-cone) axial gauge, which we already briefly encountered in section 1.4. We will see more of this in the later lectures when we study quantization of non-abelian gauge theories. For now, take this simply as a way (importantly, not the only one!!) to parametrize the two physical degrees of freedom that we are interested into.

[4] As an important comment here, notice that transversality alone is not enough to specify the two physical polarizations only, in fact any longitudinal polarization $\varepsilon^\mu_L \propto (a, 0, 0, a)$ still fulfills transversality. On the other hand, ε^μ_L cannot be properly normalized since $\varepsilon^\mu_L \varepsilon^*_{L\mu} = 0$.

We would like now to use these two momenta to build a representation for the polarization vectors using spinor-helicity variables. Let us start considering the following two "four vectors"

$$\eta_1^\mu = [r\gamma^\mu p\rangle\,, \qquad \eta_2^\mu = \langle r\gamma^\mu p]\,. \tag{2.53}$$

By direct computation, using the Dirac equation eqs. (2.20) and (2.21), we see that

$$\eta_{1,2}^\mu p_\mu = \eta_{1,2}^\mu r_\mu = 0\,. \tag{2.54}$$

Moreover, these vectors also satisfy the following properties

$$\begin{aligned}
&(\eta_1^\mu)^* = \eta_2\,,\\
&(\eta_1^\mu)^*\eta_{2,\mu} = \langle r\gamma^\mu p]\langle r\gamma_\mu p] = 0\,,\\
&(\eta_1^\mu)^*\eta_{1,\mu} = \langle r\gamma^\mu p]\langle p\gamma_\mu r] = 2\langle rp\rangle[rp] = (\eta_2^\mu)^*\eta_{2,\mu}\,,
\end{aligned} \tag{2.55}$$

where we used Fierz identities. These equations imply that these two vectors are linearly independent from r^μ and p^μ, so all together these four vectors form a basis in four-dimensional Minkowski space. Any other four-vector can be then written as a linear combination of these 4 basis vectors. This must be true, in particular, for the polarization vectors, i.e.

$$\varepsilon_\pm^\mu(p,r) = A_\pm p^\mu + B_\pm r^\mu + C_\pm \eta_1^\mu + D_\pm \eta_2^\mu\,. \tag{2.56}$$

From here, imposing conditions eq. (2.52), one immediately can see that $A_\pm = B_\pm = 0$ for both polarizations, i.e. the polarizations are transverse to the plane spanned by p, r. In addition, imposing the normalization conditions in eq. (2.50), we find that there are two possible independent choices for the two polarization vectors, each involving only one of the vectors $\eta_{1,2}^\mu$. Which one we call which is conventional. We always assume incoming momenta and we *define*

$$\begin{aligned}
\varepsilon_+^\mu(p,r) &= -\frac{[r\gamma^\mu p\rangle}{\sqrt{2}[rp]} &&= (\varepsilon_-^\mu)^*\,,\\
\varepsilon_-^\mu(p,r) &= +\frac{\langle r\gamma^\mu p]}{\sqrt{2}\langle rp\rangle} &&= (\varepsilon_+^\mu)^*\,.
\end{aligned} \tag{2.57}$$

From these equations, using Fierz identities, it's straightforward to see that these vectors are correctly normalized as required above

$$(\varepsilon_+^\mu)^*\varepsilon_{+\mu} = (\varepsilon_-^\mu)^*\varepsilon_{-\mu} = -1\,,$$

$$(\varepsilon_+^\mu)^*\varepsilon_{-\mu} = (\varepsilon_-^\mu)^*\varepsilon_{+\mu} = 0\,.$$

Little Group Scaling

With the formulas derived in section 2.1.3, it is also immediate to see how these newly defined polarization vectors transform under the little group. Consider the transformation

$p \to \Lambda p = p$ such that $|p\rangle \to z|p\rangle$ and $|p] \to 1/z|p]$. Then just by counting the number of square or angle brackets of the momentum p, it's obvious to see that

$$\varepsilon_+^\mu(p,r) \to z^2 \varepsilon_+^\mu(p,r)\,, \qquad \varepsilon_-^\mu(p,r) \to \frac{1}{z^2}\varepsilon_-^\mu(p,r)\,, \tag{2.58}$$

i.e. the spin-1 polarizations scale as "double" the spin 1/2 ones do, as expected. This shows that, as described at the beginning of this chapter, the polarization vectors carry the information on the spin of the corresponding particles. By performing the appropriate little group transformation and studying the scaling of the amplitude, we can "learn" if the particle involved is a scalar, a spin 1/2, a spin 1, etc.... It is enough to just count how many square or angle brackets there are for a given momentum, to be able to immediately extract the corresponding spin.[5]

As an interesting observation, which allows us to discuss the role of the auxiliary vector r^μ, we can ask ourselves what happens if we perform a little-group transformation that keeps this moment invariant. In this case, as it is easy to see from eq. (2.57), the same number of square and angle brackets of the momentum r^μ appear at the numerator and denominator of each expression, showing that the polarization vectors *do not scale at all* with r^μ. This hints to the fact that the dependence on this vector is somewhat non-physical. In fact, it can be shown that changing the vector r^μ does not change the physics of the scattering amplitude. In particular, it corresponds to performing a "gauge transformation" for the external photon considered, in the class of the "light-cone" axial gauges. Notice the important subtlety: this is not a gauge transformation at the Lagrangian level and it would not affect any photons propagating in the loops. It is more like a sort of change of "gauge reference frame" for the external particle, which can indeed be fixed independently for each particle in the amplitude. You will see some more examples of this in the exercises.

To demonstrate more precisely what we mean when we say that changing the vector r^μ does not change the physics, we take two polarization vectors $\varepsilon_\mu^-(p,r)$ and $\varepsilon_\mu^-(p,q)$ depending on two different gauge vectors r^μ and q^μ and compute their difference. We then have

$$\begin{aligned}
\varepsilon_\mu^-(p,r) - \varepsilon_\mu^-(p,q) &= \frac{1}{\sqrt{2}}\left[\frac{\langle r\gamma^\mu p]}{\langle rp\rangle} - \frac{\langle q\gamma^\mu p]}{\langle qp\rangle}\right] \\
&= \frac{1}{\sqrt{2}}\frac{\langle r\gamma^\mu p]\langle qp\rangle - \langle q\gamma^\mu p]\langle rp\rangle}{\langle rp\rangle\langle qp\rangle} \\
&= -\frac{1}{\sqrt{2}}\frac{\langle r\gamma^\mu \not{p} q\rangle - \langle q\gamma^\mu \not{p} r\rangle}{\langle rp\rangle\langle qp\rangle} \\
&= -\frac{1}{\sqrt{2}}\frac{\langle r(\gamma^\mu \not{p} + \not{p}\gamma^\mu) q\rangle}{\langle rp\rangle\langle qp\rangle} \\
&= -\frac{1}{\sqrt{2}}\left[\frac{2\langle rq\rangle}{\langle rp\rangle\langle qp\rangle}\right] p^\mu\,,
\end{aligned} \tag{2.59}$$

where we used $\gamma^\mu \not{p} + \not{p}\gamma^\mu = p^\nu\{\gamma^\mu,\gamma^\nu\} = 2p^\mu$. We see that the difference of the two polarization vectors is proportional to the boson momentum p^μ. Now recall that polarization vectors

[5] A similar construction can be generalized also for spin 2, but we won't consider it here.

only appear in the amplitude as $\varepsilon_\mu \mathcal{M}^\mu$ and the Ward identity in QED (a statement of gauge invariance) states that[6]

$$p_\mu \mathcal{M}^\mu = 0\,. \tag{2.60}$$

This shows that changing the arbitrary vector r^μ corresponds to a gauge transformation and, through the Ward identities, does not affect the final result for the scattering amplitude.

[6]We stress here that in a general gauge theory, as we will see later on, this statement is strictly true only if all other external bosons are on-shell. In QED this is not necessary.