Advanced Quantum Field Theory SS 2025

Lecturer: Prof. Lorenzo Tancredi
Assistants: Federico Buccioni, Sara Ditsch, Felix Forner, Chiara Savoini
https://www.groups.ph.tum.de/ttpmath/teaching/ss-2025/

Sheet 04: Yang-Mills classical field theory, EOMs and Noether's theorem
To be handed in to your tutors by Friday, May 23rd

Problem 1 - Non-abelian equations of motions with scalar fields

Consider a non-abelian gauge theory where a gauge field $\mathbf{A}_\mu = A^a_\mu T^a$ couples to scalar fields $\phi, \phi^\dagger$, in the fundamental representation of the gauge group. The Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_A + (D_\mu \phi)^\dagger (D^\mu \phi) - m^2 \phi^\dagger \phi - \lambda \left(\phi^\dagger \phi\right)^2 , \tag{1}$$

where the kinetic term for the gauge field is

$$\mathcal{L}_A = \frac{1}{2g^2} \operatorname{Tr}[\mathbf{F}^{\mu\nu} \mathbf{F}_{\mu\nu}] \tag{2}$$

with

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + [\mathbf{A}_\mu, \mathbf{A}_\nu] = -ig F^a_{\mu\nu} T^a . \tag{3}$$

In the previous equation, g is a constant and represents the coupling, and the index a is in the adjoint representation of the gauge group.

1. Show that the equations of motion for the gauge field can be written in a form

$$(D^\mu F_{\mu\nu})^a = g J^a_\nu . \tag{4}$$

   Find the expression of the current J^a_ν.
   Note that in Eq.(4) the covariant derivative acts on $\mathbf{F}^{\mu\nu}$ which has values in the Lie Algebra and transforms in the adjoint representation!
   Hint: For a generic field $\chi = \chi^a T^a$ in the adjoint representation, the covariant derivative acts as

$$D_\mu \chi^a = \partial_\mu \chi^a + g f^{abc} A^b_\mu \chi^c ,$$

   where f^{abc} are the structure constants.

2. Show that $D_\mu D_\nu \mathbf{F}^{\mu\nu} = 0$. Then, write down the conservation equation for the current J^a_μ.

3. Find the equations of motion for the scalar field ϕ.

Problem 2 - Equations of motions with spinor fields

Consider the Yang-Mills Lagrangian coupled to a Dirac field

$$\mathcal{L}_{YM} = \overline{\psi}(i\not{D} - m)\psi + \frac{1}{2g^2} \operatorname{Tr}[\mathbf{F}^{\mu\nu} \mathbf{F}_{\mu\nu}] , \tag{5}$$

with $\mathbf{F}_{\mu\nu}$ defined in Eq.(3) where T^a are the generators of $SU(N)$ in the fundamental representation.

1. Derive the equations of motion that were quoted in the lectures, namely

$$\begin{aligned} \partial^\mu F^a_{\mu\nu} + g f^{abc} A^{b,\mu} F^c_{\mu\nu} &= -g T^a_{ij} \overline{\psi}_i \gamma^\nu \psi_j \\ (i\partial\!\!\!/ - m)\psi_i &= -g A\!\!\!/^a T^a_{ij} \psi_j \,. \end{aligned} \tag{6}$$

2. Show that the conserved current under global $SU(N)$ transformations is

$$J^a_\mu = -\overline{\psi}_i \gamma^\mu T^a_{ij} \psi_j + f^{abc} A^b_\nu F^c_{\mu\nu} \,, \qquad \text{with} \quad \partial_\mu J^a_\mu = 0 \,. \tag{7}$$

3. Prove that the matter current
$$j^a_\mu = -\overline{\psi} \gamma_\mu T^a_{ij} \psi_j$$
is gauge covariant, i.e. show that
$$D^\mu j^a_\mu = 0 \,.$$

Problem 3 - Additional term in the non-abelian Lagrangian

For this exercise we could consider a <u>non-abelian</u> gauge field theory with an arbitrary gauge group. For definiteness, let us take the explicit case of $SU(N)$. The Lagrangian density is given by

$$\mathcal{L}_A = \frac{1}{2g^2} \mathrm{Tr}\big[\mathbf{F}_{\mu\nu} \mathbf{F}^{\mu\nu}\big] = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu,\,a} \tag{8}$$

with $\mathbf{F}_{\mu\nu}$ defined in Eq.(3). At first sight, it appears possible to add the term

$$\delta\mathcal{L} = \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left[\mathbf{F}^{\mu\nu} \mathbf{F}^{\rho\sigma}\right] \tag{9}$$

to the standard Lagrangian.

1. Discuss how you expect this term to change under a parity transformation.
2. Show that this term is, in fact, a total derivative $\delta\mathcal{L} = \partial_\mu J^\mu$.
3. Find the expression for the vector current J^μ and explain why such terms do not change the equations of motion.
4. Repeat the same exercise in the <u>abelian</u> case. What changes in the corresponding current J^μ?

Problem 4 - Noether's theorem

Consider the Lagrangian density
$$\mathcal{L} = \partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2} m^2 \phi_a \phi_a$$
for a triplet of real fields ϕ_a, where $a \in \{1, 2, 3\}$.

1. Show that the Lagrangian is invariant under the infinitesimal $SO(3)$ rotation by θ

$$\phi_a \to \phi_a + \theta \epsilon_{abc} \eta_b \phi_c$$

where η_a is a unit vector.

2. Compute the Noether current j^μ associated with this rotation. Deduce that the quantities

$$Q_a = \int d^3x \epsilon_{abc} \dot{\phi}_b \phi_c$$

are all conserved. Check this explicitly by using the equations of motion.