**Advanced Quantum Field Theory SS 2025**

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https://www.groups.ph.tum.de/ttpmath/teaching/ss-2025/

**Sheet 03: Group Theory**
To be handed in to your tutors by Friday, May 16th

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# 1 su(N) and the anomaly coefficients

We refer to the $\mathbf{su}(N)$ algebra generators in a given representation $R$ as $T^a_R$, $a = 1, ..., d(\mathbf{su}(N))$, where $d(\mathbf{su}(N))$ is the dimension of the algebra. If no subscript is present, then the fundamental representation is assumed. The totally symmetric invariant $d^{abc}$ is defined as

$$d^{abc} = 2\, tr\Big(T^a\{T^b, T^c\}\Big). \tag{1}$$

The generalisation to an arbitrary representation $R$ is given by

$$tr\Big(T^a_R\{T^b_R, T^c_R\}\Big) = \frac{1}{2}A(R)\, d^{abc} = A(R)\, tr\Big(T^a\{T^b, T^c\}\Big), \tag{2}$$

where the constant $A(R)$ is called *anomaly coefficient*. In the fundamental representation, $A(R) = 1$.

1. Consider a representation $R$ and its complex conjugated representation $\bar{R}$.
   Show that $A(R) = -A(\bar{R})$ and argue that for a real representation $A(R) = 0$.
   (**Hint**: it is useful to first prove that $A(R)d^{abc}$ is real.)

2. Any reducible representation of a Lie algebra can be decomposed as the direct sum of irreducible representations. Show that if
   $$R = R_1 \oplus R_2$$
   then
   $$A(R) = A(R_1 \oplus R_2) = A(R_1) + A(R_2). \tag{3}$$
   Remember that the direct sum of two matrices $A$ and $B$ is the matrix $C$ in block diagonal form, with $A$ and $B$ being the two blocks.

3. Show that, for a tensor product of representations, the analogous formula is
   $$A(R_1 \otimes R_2) = A(R_1)\, d(R_2) + d(R_1)\, A(R_2), \tag{4}$$
   where $d(R_i)$ is the dimension of the representation $R_i$.
   Remember that the tensor product of two matrices, $A$ and $B$, of dimension $m$ and $n$ respectively, is defined as the matrix $C$ with entries $C_{ij,lk} = A_{il}B_{jk}$. First prove that
   $$(A \otimes B)(A' \otimes B') = (AA') \otimes (BB')\,, \tag{5}$$
   $\forall A, A'$ $m \times m$ matrices and $B, B'$ $n \times n$ matrices. Then argue that
   $$tr(A \otimes B) = tr(A)tr(B). \tag{6}$$

4. What is the anomaly coefficient $A(\mathbf{10})$ for $SU(4)$? Use the fact that $\mathbf{4} \otimes \mathbf{4} = \mathbf{6} \oplus \mathbf{10}$ and that $\mathbf{6}$ is a real representation.

## 2 The trace scalar product

It can be proved that a Lie algebra is compact if and only if there exists a positive-definite scalar product, which is invariant under the adjoint action of the group. Namely, the following conditions are true

$$(A, B) = (Ad(g)A, Ad(g)B) \quad \forall g \in G, \tag{7}$$

and

$$(A, A) \geq 0. \tag{8}$$

For a matrix group, the scalar product is the trace

$$(A, B) = -Tr(A, B). \tag{9}$$

1. Prove that Eq.(9) is invariant under the action of the adjoint representation. The non-trivial part of the theorem is proving the positive definiteness, which we will not do here.

2. Verify that Eq.(8) holds for $su(2)$ but not for $gl(2, \mathbb{C})$.

## 3 Vector bosons self interaction in $SU(N)$

Working in QED, consider the scattering process

$$q(p_1)\, \bar{q}(p_2) \to \gamma(k_1)\, \gamma(k_2)\,, \tag{10}$$

where $\gamma$ is a photon and the particles $q(\bar{q})$ are massless fermions.

1. Write down the scattering amplitude $\mathcal{M}$ for this process and check gauge invariance by showing that the Ward identity is satisfied, namely

$$\mathcal{M} = \mathcal{M}^{\mu\nu} \epsilon^*_\mu(k_1) \epsilon^*_\nu(k_2) \quad \Rightarrow \quad \mathcal{M}^{\mu\nu} k_{1,\mu}\, \epsilon^*_\nu(k_2) = \mathcal{M}^{\mu\nu} \epsilon^*_\mu(k_1)\, k_{2,\nu} = 0\,, \tag{11}$$

where $\epsilon^*_\mu(k_i)$ are the polarisation vectors of the final-state photons. Is the Ward identity satisfied Feynman diagram by Feynman diagram or when all of them are summed together?

Let us try to generalise QED, which is a gauge theory with an abelian $U(1)$ symmetry group, to a more general $SU(N)$ gauge theory. In this scenario, the fermions $q$ and $\bar{q}$ carry a non-Abelian charge and transform under a given representation $R$ and $\bar{R}$ respectively. For simplicity, let us decide that $R$ is the fundamental representation. Then, the photon generalises to a particle that we label with $g$.

2. Assume that $g$, which has spin-1, is also produced in a $q\bar{q}$ annihilation. To which irreducible representations of $SU(N)$ should $g$ belong?
For solving the remaining part of the exercise, assume that $g$ belongs to the non-trivial representation.

3. Starting from the QED $q\bar{q} \to \gamma$ interaction vertex, argue why the simplest generalisation to $q\bar{q} \to g$ reads
$$V(q_j\bar{q}_l \to g^{a,\mu}) = i\,\lambda\,T^a_{lj}\gamma^\mu, \tag{12}$$
where $\lambda$ is a coupling constant, $j\,(l) \in \{1, \ldots, N\}$ is the group index carried by $q\,(\bar{q})$, and $T^a$ is a $SU(N)$ generator in the fundamental representation which satisfies the algebra
$$\left[T^a, T^b\right] = i f^{abc} T^c\,, \tag{13}$$
for a fixed set of structure constants $f^{abc}$, which uniquely characterise the algebra.

4. Using Eq.(12), compute the same amplitudes of **point 1.**, where now $\gamma(k_1)$ and $\gamma(k_2)$ are replaced by $g(k_1)$ and $g(k_2)$. The latter two carry $SU(N)$ adjoint indices $a$ and $b$, respectively. Show that now the Ward identity is not fulfilled any longer, and in particular that
$$k_{1,\mu}\,\mathcal{M}^{\mu\nu}\epsilon^*_\nu(k_2) = \lambda^2 f^{abc}\,T^c_{lj}\,\bar{v}(p_2)\not{\epsilon}(k_2)u(p_1)\,. \tag{14}$$

5. Upon rewriting Eq.(14) as
$$k_{1,\mu}\,\mathcal{M}^{\mu\nu}\epsilon^*_\nu(k_2) = \left(-i\lambda f^{abc}\right) \times \left(i\,\lambda\,T^c_{lj}\,\bar{v}(p_2)\gamma^\alpha u(p_1)\right) \times \epsilon^*_\alpha(k_2), \tag{15}$$
one can see that the second term in bracket has the right structure of a $q\bar{q}g$ interaction vertex. This suggests that there could be a third Feynman diagram which would produce Eq.(14) and restore the Ward identity. It is reasonable to think that the particle $g$ can interact with itself. Assuming this, draw the extra Feynman diagram with the appropriate momentum flow.

6. Considering that $g$ is a spin-1 massless particle, show that this additional Feynman diagram has to have the general form
$$i\,\lambda T^c_{lj}\,\bar{v}(p_2)\gamma^\alpha u(p_1) \times \left(\frac{-i}{k^2}\right)\lambda\, f^{abc} V_{\alpha\mu\nu}(k, -k_1, -k_2) \times \epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2), \tag{16}$$
where $k = k_1 + k_2$.[1]

7. We now want to find an explicit expression for $V_{\alpha\mu\nu}$. Using *dimensional analysis*, *Lorentz covariance* and *Bose symmetry*, discuss what is the most general dependence of $V_{\alpha\mu\nu}$ on the momenta $k$, $k_1$ and $k_2$. Finally, provide a unique Lorentz structure for this 3-$g$ interaction vertex and fix the overall unknown factor by imposing *gauge invariance* at amplitude level.

[1]In $V_{\alpha\mu\nu}(p_i, p_j, p_k)$ we adopt the convention where all momenta are incoming.