Advanced Quantum Field Theory SS 2025

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https://www.groups.ph.tum.de/ttpmath/teaching/ss-2025/

Sheet 2: Spinor-Helicity Formalism
To be handed in to your tutors by Friday, May 9th

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1 Explicit Representations

Remember that

$$(\sigma^\mu)_{a\dot{b}} = \left(1, \sigma^i\right)_{a\dot{b}}\,, \quad (\bar{\sigma}^\mu)^{\dot{a}b} = \left(1, -\sigma^i\right)^{\dot{a}b},$$

with Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Furthermore, the 2-component Levi-Civita symbol is defined as

$$\varepsilon^{ab} = \varepsilon^{\dot{a}\dot{b}} = -\varepsilon_{ab} = -\varepsilon_{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

1.1 Explicit Spinor Representations

Consider the momentum four-vector,

$$p^\mu = (E, E\sin\theta\cos\phi, E\sin\theta\sin\phi, E\cos\theta)$$

and address the following points:

1. Express $p_{a\dot{b}}$ and $p^{\dot{a}b}$ in terms of $E, \sin\frac{\theta}{2}, \cos\frac{\theta}{2}$ and $e^{\pm i\phi}$.
2. Show that the helicity spinor $|p\rangle^{\dot{a}} = \sqrt{2E}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}$ solves the massless Weyl equation.
3. Find the expressions for the spinors $\langle p|_{\dot{a}}\,, |\,p]_a$, and $[p|^a$ and check that they satisfy $p_{a\dot{b}} = |p]_a\, \langle p|_{\dot{b}}$ and $p^{\dot{a}b} = |p\rangle^{\dot{a}}\,[p|^b$.

1.2 Explicit Representation Polarization Vector

In this exercise, we establish the connection between the polarization vector for a massless spin-1 boson of momentum p^μ

$$\epsilon_-^\mu(p;q) = \frac{\langle q\,|\gamma^\mu|\,p]}{\sqrt{2}\langle qp\rangle} \tag{1}$$

and the more familiar representation

$$\tilde{\epsilon}_-^\mu(p) = \frac{e^{-i\phi}}{\sqrt{2}}(0, \cos\theta\cos\phi + i\sin\phi, \cos\theta\sin\phi - i\cos\phi, -\sin\theta). \tag{2}$$

Note that for $\theta = \phi = 0$, we have $\tilde{\epsilon}_-^\mu(p) = \frac{1}{\sqrt{2}}(0, 1, -i, 0)$.

1. Since $\tilde{\epsilon}_-^\mu(p)$ is null, $(\tilde{\epsilon}_-^\mu(p))_{a\dot{b}} = (\sigma_\mu)_{a\dot{b}}\,\tilde{\epsilon}_-^\mu(p)$ can be written as a product of a square and an angle spinors. To see this specifically, calculate $(\tilde{\epsilon}_-^\mu(p))_{a\dot{b}}$ and then find an angle spinor $\langle r|$ such that $(\tilde{\epsilon}_-^\mu(p))_{a\dot{b}} = |p]_a\,\langle r|_{\dot{b}}$. Verify that $\langle pr\rangle = -\sqrt{2}$.

2. Show that from Eq. (1) it follows that $(\epsilon_-(p;q))_{a\dot{b}} = \frac{\sqrt{2}}{\langle qp\rangle}\,|p]\,\langle q|$.

3. Suppose there is a constant c_- such that $\epsilon_-^\mu(p;q) = \tilde{\epsilon}_-^\mu(p) + c_- p^\mu$. Show that this relation requires $\langle rp\rangle = \sqrt{2}$ and $c_- = -\langle rq\rangle/\langle pq\rangle$.

1.3 Spinor identities

In this exercise we prove some of the identities among spinor products which were discussed in the lecture:

1. **Fierz identity:**
$$[p\gamma^\mu q\rangle\,[k\gamma_\mu l\rangle = 2[pk]\langle lq\rangle$$
$$\langle p\gamma^\mu q]\,\langle k\gamma_\mu l] = 2\langle pk\rangle[lq]$$

2. **Reversal identity:**
$$\langle p\gamma^{\mu_1}\ldots\gamma^{\mu_{2n}}q\rangle = -\,\langle q\gamma^{\mu_{2n}}\ldots\gamma^{\mu_1}p\rangle \quad \text{for an even number of Dirac matrices}$$
$$\langle p\gamma^{\mu_1}\ldots\gamma^{\mu_{2n+1}}q] = [q\gamma^{\mu_{2n+1}}\ldots\gamma^{\mu_1}p\rangle \quad \text{for an odd number of Dirac matrices}$$

2 QED Compton Scattering

In this exercise, we compute the tree-level amplitude for electron-positron annihilation $\mathcal{A}(e^+e^- \to \gamma\gamma)$ using spinor-helicity formalism. We assume that the fermions are massless and take all particles as incoming, i.e.
$$e^+(p_1) + e^-(p_2) + \gamma(p_3) + \gamma(p_4) \longrightarrow 0\,,$$
with $\sum_{i=1}^4 p_i^\mu = 0$.

1. Which are the independent helicity configurations?

2. Draw the diagrams contributing to this process, insert the QED Feynman rules and perform the algebra using the spinor-helicity formalism.

3. Exploit the helicity amplitudes computed in the previous point to check that $\sum\limits_{\text{spins}} |\mathcal{A}(\text{spins})|^2$ gives the well-known result
$$\sum_{\text{spins}} |\mathcal{A}(\text{spins})|^2 = 8e^4\left(\frac{u}{t} + \frac{t}{u}\right) \qquad (3)$$
where $u = 2p_1\cdot p_4$ and $t = 2p_1\cdot p_3$, obtained from the trace formalism à la Peskin & Schroeder (see Eq. (5.87) of the corresponding book).