Advanced Quantum Field Theory SS 2025

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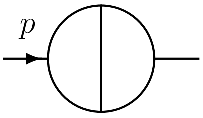
https://www.groups.ph.tum.de/ttpmath/teaching/ss-2025/

Sheet 01: Loop integrals and dispersion relations
To be handed in to your tutors by Friday, May 2nd

Problem 1 - A two-loop massless bubble

The goal of this exercise is to get some familiarity with dimensional regularisation and with the idea of Integration-By-Parts identities (IBPs). We will use them to calculate analytically the following two-loop integral in dimensional regularisation ($d = 4 - 2\epsilon$)

$$\mathcal{I}(p^2) =$$



$$= \int \frac{d^d q_1}{(2\pi)^d} \int \frac{d^d q_2}{(2\pi)^d} \, \frac{1}{q_1^2 \, (q_1 - p)^2 \, (q_1 - q_2)^2 \, q_2^2 \, (q_2 - p)^2} \,. \tag{1}$$

This integral is finite for $\epsilon = 0$ and is given by the very simple expression

$$\mathcal{I}(p^2) = -\frac{6\,\zeta_3}{(4\,\pi)^4 \, p^2} \; + \mathcal{O}(\epsilon) \,,$$

where $\zeta_3 = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.20206...$ is the Riemann zeta function. However, deriving this result by means of direct integration is difficult. IBPs provide instead a much more elegant way.

1. Start off by performing a Wick rotation

$$q_1^0 = -i\, k^0 \,, \qquad q_2^0 = -i\, l^0 \,, \qquad p_0 = -i\, p_E^0 \tag{2}$$

   in order to go to the Euclidean region, where the integral becomes

$$\mathcal{I}(p^2) = \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} \, \frac{1}{k^2 \, (k - p_E)^2 \, (k - l)^2 \, l^2 \, (l - p_E)^2} = \mathcal{I}_E(p_E^2) \,. \tag{3}$$

   Here, the vectors $k,\ l,\ p_E$ are Euclidean, i.e. $k^2 = k_0^2 + \vec{k}^2$, $l^2 = l_0^2 + \vec{l}^2$ and $p_E^2 = -p^2$.

2. Let us now focus on the Euclidean integral $\mathcal{I}_E(p_E^2)$. Argue why, in dimensional regularisation, we can write

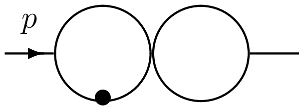
$$\int \frac{d^d k}{(2\pi)^d} \int \frac{d^d l}{(2\pi)^d} \, \left[\frac{\partial}{\partial k_\mu} v_\mu \frac{1}{k^2 \, (k - p_E)^2 \, (k - l)^2 \, l^2 \, (l - p_E)^2} \right] = 0 \,, \tag{4}$$

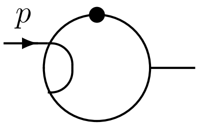
   for every vector $v_\mu = k_\mu, l_\mu, p_{E,\mu}$. This type of relations are referred to as Integration-By-Parts Identities, or IBPs.

3. (**Bonus**) Specialize the IBP above by choosing $v_\mu = k_\mu - l_\mu$ and use it to prove that the integral $\mathcal{I}_E(p_E^2)$ can be *"reduced"* to

$$\mathcal{I}_E(p_E^2) = \frac{2}{d-4}\left(\mathcal{I}_1(p_E^2) - \mathcal{I}_2(p_E^2)\right)\,, \tag{5}$$

where:

$$\mathcal{I}_1(p_E^2) =$$

$$= \int \frac{d^dk}{(2\pi)^d}\int \frac{d^dl}{(2\pi)^d}\,\frac{1}{k^4\,(k-p_E)^2\,l^2\,(l-p_E)^2}\,, \tag{6}$$

$$\mathcal{I}_2(p_E^2) =$$

$$= \int \frac{d^dk}{(2\pi)^d}\int \frac{d^dl}{(2\pi)^d}\,\frac{1}{k^4\,(k-p_E)^2\,(k-l)^2\,(l-p_E)^2}\,. \tag{7}$$

The black dots on the graphs indicate squared propagators, as you can see from the momentum representation of the integrals.

4. To compute $\mathcal{I}_1(p_E^2)$ and $\mathcal{I}_2(p_E^2)$, start off by defining the Euclidean one-loop bubble with arbitrary powers of the propagators

$$\mathcal{B}(q_E^2;a,b) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^a\;((k-q_E)^2)^b}\,. \tag{8}$$

Using Feynman parameters prove that

$$\mathcal{B}(q_E^2;a,b) = \frac{(4\pi)^\epsilon}{16\pi^2}\,\frac{\Gamma(2-\epsilon-a)\,\Gamma(2-\epsilon-b)\,\Gamma(a+b-2+\epsilon)}{\Gamma(a)\,\Gamma(b)\,\Gamma(4-2\epsilon-a-b)}\,\left(q_E^2\right)^{2-\epsilon-a-b}\,, \tag{9}$$

where as usual $d = 4-2\epsilon$.

5. Using only Eq. (9) and defining

$$S_\epsilon = \frac{(4\pi)^\epsilon\,\Gamma(1+\epsilon)\,\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\,, \tag{10}$$

prove that:[1]

$$\mathcal{I}_1(p_E^2) = \left(\frac{S_\epsilon}{16\pi^2}\right)^2\left(-\frac{1}{\epsilon^2(1-2\epsilon)}\right)\left(p_E^2\right)^{-1-2\epsilon}\,, \tag{11}$$

$$\mathcal{I}_2(p_E^2) = \left(\frac{S_\epsilon}{16\pi^2}\right)^2\left(-\frac{1}{\epsilon^2(1-2\epsilon)}\right)\frac{\Gamma(1-2\epsilon)^2\,\Gamma(1+2\epsilon)}{\Gamma(1-\epsilon)\,\Gamma(1+\epsilon)^2\,\Gamma(1-3\epsilon)}\left(p_E^2\right)^{-1-2\epsilon}\,. \tag{12}$$

[1] Make use, where necessary, of the functional identity $\Gamma(1+x) = x\,\Gamma(x)$ in order to extract explicitly all poles in $1/\epsilon$.

6. (**Bonus**) Using the series expansion

$$\Gamma(1+n\epsilon)\, e^{n\,\gamma\,\epsilon} = 1 + \frac{\pi^2}{12} n^2 \epsilon^2 - \frac{\zeta_3}{3} n^3 \epsilon^3 + \mathcal{O}(\epsilon^4)\,,$$

where γ is the Euler-Mascheroni constant, expand all Γ functions around $\epsilon = 0$ and prove that

$$\frac{\Gamma(1-2\epsilon)^2\,\Gamma(1+2\epsilon)}{\Gamma(1-\epsilon)\,\Gamma(1+\epsilon)^2\,\Gamma(1-3\epsilon)} = 1 - 6\,\zeta_3\,\epsilon^3 + \mathcal{O}(\epsilon^4)\,. \tag{13}$$

7. Finally putting everything together show that

$$\mathcal{I}_E(p_E^2) = \left(\frac{S_\epsilon}{16\pi^2}\right)^2 (\,6\,\zeta_3 + \mathcal{O}(\epsilon))\,\left(p_E^2\right)^{-1-2\epsilon} \tag{14}$$

such that in the Minkowskian, physical, region we have

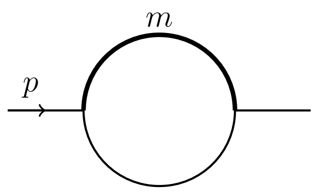
$$\mathcal{I}(p^2) = \left(\frac{S_\epsilon}{16\pi^2}\right)^2 (\,6\,\zeta_3 + \mathcal{O}(\epsilon))\,\left(-p^2 - i\,\delta\,\right)^{-1-2\epsilon} = -\frac{6\,\zeta_3}{(4\,\pi)^4\,p^2} + \mathcal{O}(\epsilon)\,, \tag{15}$$

where $0 < \delta \ll 1$ comes from Feynman causal prescription.

Problem 2 - Phase-space integrals and dispersion relations

Consider the following one-loop bubble

$$\mathcal{B}(s,m) =$$



$$= \int \frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2((k-p)^2 - m^2)}\,, \tag{16}$$

where $p^2 \equiv s$ and the thick line represents a massive propagator with mass m.

The goal of this exercise is to exploit the connection between the imaginary part of the one-loop diagram and the corresponding 2-body phase-space integral, through the unitarity relation established by the *optical theorem*. The optical theorem states that the imaginary part of a forward diagram is the sum of the contributions arising from all possible intermediate states going on-shell. This effectively means to consider all the unitary cuts across the propagators. In this case, there is only one way to cut the loop diagram, which requires $s \geq m^2$.

1. Compute, in generic d-dimensions, the 2-body phase-space integral

$$\mathcal{R}_2^{(d)}(p_1, p_2; s) = \int \frac{d^d p_1}{(2\pi)^{d-1}} \delta^+(p_1^2) \int \frac{d^d p_2}{(2\pi)^{d-1}} \delta^+(p_2^2 - m^2)\, (2\pi)^d \delta^{(d)}(p - p_1 - p_2)\,, \tag{17}$$

associated with the decay of a particle of momentum p into a massless and a massive particles with momenta p_1 and p_2 respectively.
Show that the result is

$$\mathcal{R}_2^{(d)}(p_1, p_2; s) = \frac{4^{2-d}}{\pi^{(d-3)/2}\Gamma\left(\frac{d-1}{2}\right)} \frac{(s-m^2)^{d-3}}{s^{(d-2)/2}}\,, \tag{18}$$

which corresponds to twice the imaginary part of the one-loop bubble in Fig.(16).

2. Taking advantage of the result in Eq.(18), argue that a d-dimensional dispersion relation for the one-loop bubble in Fig.(16) can be written as

$$\mathcal{B}^{(d)}(s, m) = \frac{2^{3-2d}}{\pi^{(d-1)/2}\,\Gamma\left(\frac{d-1}{2}\right)} \int_{m^2}^{\infty} dt\, \frac{(t-m^2)^{d-3}}{t^{(d-2)/2}\,(t-s)} \qquad \forall s < m^2\,. \tag{19}$$

Specialize it for $d = 2$ and $d = 4$, and discuss the (lack of) convergence of the corresponding integrals. What is the physical origin of the divergence in the two cases?

3. Let us focus on the $d = 4$ case now. The dispersion relation can be rendered finite by performing a *subtraction*. Explicitly, instead of computing $\mathcal{B}^{(d=4)}(s, m)$, which is divergent, write a dispersion relation for the difference

$$\overline{\mathcal{B}}^{(d=4)}(s, m) = \mathcal{B}^{(d=4-2\epsilon)}(s, m) - \mathcal{B}^{(d=4-2\epsilon)}(0, m)\,. \tag{20}$$

Show that this dispersion relation is finite for $\epsilon = 0$ and that $\forall s < m^2$, it holds that

$$\overline{\mathcal{B}}^{(d=4)}(s, m) = \frac{1}{16\pi^2}\left\{1 + \frac{(m^2 - s)}{s}\log\left(\frac{m^2 - s}{m^2}\right)\right\}\,. \tag{21}$$

4. Would you know how to compute $\mathcal{B}^{(d)}(0, m)$ for any value of d and expand it close to $d = 4 - 2\epsilon$? If you do so, we can recover the complete result for the bubble in $d = 4 - 2\epsilon$ as

$$\begin{aligned}\mathcal{B}^{(d=4-2\epsilon)}(s, m) &= \overline{\mathcal{B}}^{(d=4)}(s, m) + \mathcal{B}^{(d=4-2\epsilon)}(0, m) \\ &= \left(\frac{C_\epsilon}{16\pi^2}\right)(m^2)^{-\epsilon}\left[\frac{1}{\epsilon} + 2 + \left(\frac{m^2}{s} - 1\right)\log\left(\frac{m^2 - s}{m^2}\right) + \mathcal{O}(\epsilon)\right]\,, \end{aligned} \tag{22}$$

where we introduced the overall normalisation $C_\epsilon = (4\pi)^\epsilon\Gamma(1+\epsilon)$ and we kept the explicit dependence on ϵ since the two corresponding integrals develop poles for $\epsilon \to 0$. What is the physical interpretation of the subtraction at $s = 0$?

5. (**Bonus**) By exploiting again the result of the d-dim 2-body phase space in Eq.(18), derive iteratively the expression of the 3-body phase space associated with the decay of a particle of momentum p into two massless and one massive particles with momenta p_1, p_2 and p_3, respectively.

   Prove that the result is

$$\mathcal{R}_3^{(d)}(p_1, p_2, p_3; s) = \frac{2^{7-4d}}{\pi^{d-2}} \frac{s^{(2-d)/2}}{\Gamma\left(\frac{d-1}{2}\right)^2} \int_{m^2}^{s} dt \frac{(s-t)^{d-3}(t-m^2)^{d-3}}{t^{(d-2)/2}}, \tag{23}$$

   where $s \equiv p^2 = (p_1 + p_2 + p_3)^2$.

   Solve explicitly the integral in $d = 4$, showing that

$$\mathcal{R}_3^{(d=4)}(p_1, p_2, p_3; s) = \frac{1}{256\pi^3 s} \left\{ s^2 - m^4 + 2m^2 s \log \frac{m^2}{s} \right\}. \tag{24}$$