9- Iterated Integrals -

GENERAL THEORY-

SS 2022

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GENERAL THEORY OF ITERATED INTEGRALS

We have seen that Feynman Integrals fulfil Differential Equations and, from their form, that the natural language to write their solution is that of ITERATED INTEGRALS.

In general, consider a space X, with coordinates $\xi = (\xi_1, \dots, \xi_s)$.

(piecewise smooth!)

Let γ be a path on X; $\gamma : [0,1] \to X$

Let $\omega_1, \dots, \omega_n$ be one-forms on X with

Define iterated integral

$$\int_\gamma \omega_1 \dots \omega_n = \int_{0 \leq t_1 \leq \dots \leq t_n \leq 1} f_1(t_1)\, dt_1 \dots f_n(t_n)\, dt_n$$

with $\int_\gamma [\text{empty}] = 1$ by def.

where

- $f_i : \mathbb{C} \to \mathbb{C}$ are complex functions defined as the pull-backs of the ω_i along the path γ

$$\gamma^* \omega_i = dt_i \, f_i(t_i)$$

$$\gamma^* \omega_i = \omega_i \circ \gamma \qquad \text{composition formula}$$

$\Rightarrow$ it just means that we are evaluating the differential one-form on a specific coordinate choice along the curve

We call the differential forms ω_i = LETTERS

the set of all letters ω_i is the ALPHABET

a combination $\omega_1 \dots \omega_n$ called a WORD

n is called the LENGTH of the iterated int

NOTE Inverse order of the FORMS

$$\int_\gamma \omega_1 \dots \omega_n = \int_{0 \le t_1 \le \dots \le t_n \le 1} f_1(t_1)\, dt_1 \dots f_n(t_n)\, dt_n$$

$$= \int_0^1 dt_n\, f_n(t_n) \int_0^{t_{n-1}} dt_{n-1}\, f_{n-1}(t_{n-1}) \dots \int_0^{t_2} dt_1\, f_1(t_1)$$

Simplest case is the ordinary LINE INTEGRAL

$$\int_\gamma \omega_1 = \int_{[0,1]} \gamma^* \omega_1 = \int_0^1 \underline{f_1(t_1)\, dt_1}$$

↑
standard
notation

[2.2]

PROPERTIES (Ex sheet 5 !)

- iterated integrals don't depend on parametrization of the path γ

- Integration is linear

$$\int_\gamma (\alpha \omega_1 \cdots \omega_n + \beta \omega_1' \cdots \omega_m') = \alpha \int_\gamma \omega_1 \cdots \omega_n + \beta \int_\gamma \omega_1' \cdots \omega_m'$$

- if $\gamma^{-1}(t) = \gamma(1-t)$ denotes the reversal of path γ

$$\int_{\gamma^{-1}} \omega_1 \cdots \omega_n = (-1)^n \int_\gamma \omega_n \cdots \omega_1$$

- PATH COMPOSITION $\gamma_1, \gamma_2 : I \to X$ two paths such that $\gamma_1(1) = \gamma_2(0)$

$$\gamma_1 \gamma_2 = \gamma_1 \circ \gamma_2 \quad \text{(first } \gamma_1 \text{ then } \gamma_2 \text{ !)}$$

then we have the formula:

$$\int_{\gamma_1\gamma_2} \omega_1 \dots \omega_n = \sum_{k=0}^{n} \int_{\gamma_1} \omega_1 \dots \omega_k \int_{\gamma_2} \omega_{k+1} \dots \omega_n$$

- Iterated integrals satisfy SHUFFLE PRODUCT

$$\int_\gamma \omega_1 \dots \omega_n \int_\gamma \omega_1' \dots \omega_m' = \int_\gamma (\omega_1 \dots \omega_n) ш (\omega_1' \dots \omega_m')$$

where recursively

$$(\omega_1 \dots \omega_n) ш (\omega_1' \dots \omega_m') = \omega_1 \left[(\omega_2 \dots \omega_n) ш (\omega_1' \dots \omega_m') \right] + \omega_1' \left[(\omega_1 \dots \omega_n) ш (\omega_2' \dots \omega_m') \right]$$

with $() ш \omega_1 \dots \omega_n = \omega_1 \dots \omega_n ш () = \omega_1 \dots \omega_n$

for example (to be interpreted under $\int$ sign!)

$$w_1 w_2 \sqcup\!\sqcup\, w_4 w_5 = w_1 (w_2 \sqcup\!\sqcup\, w_4 w_5) + w_4 (w_1 w_2 \sqcup\!\sqcup\, w_5)$$

empty $\int_\gamma ( ) = 1$!

$$= w_1 w_2 (( ) \sqcup\!\sqcup\, w_4 w_5) + w_1 w_4 (w_2 \sqcup\!\sqcup\, w_5)$$

$$+ w_4 w_1 (w_2 \sqcup\!\sqcup\, w_5) + w_4 w_5 (w_1 w_2 \sqcup\!\sqcup\, ( ))$$

$$= w_1 w_2 w_4 w_5 + w_1 w_4 w_2 [( ) \sqcup\!\sqcup\, w_5]$$

$$+ w_1 w_4 w_5 (w_2 \sqcup\!\sqcup\, ( )) + w_4 w_1 w_2 (( ) \sqcup\!\sqcup\, w_5)$$

$$+ w_4 w_1 w_5 (w_2 \sqcup\!\sqcup\, ( )) + w_4 w_5 w_1 w_2$$

$$= w_1 w_2 w_4 w_5 + w_1 w_4 w_2 w_5 + w_1 w_4 w_5 w_2$$

$$+ w_4 w_1 w_2 w_5 + w_4 w_1 w_5 w_2 + w_4 w_5 w_1 w_2$$

$\Rightarrow$ all combinations that preserve relative order of two words!

Mixing two decks of cards, preserving relative order!

HOMOTOPY INVARIANCE

two paths $\gamma_1, \gamma_2 : [0,1] \to X$, X manifold,

with the same end points $\gamma_1(0) = \gamma_2(0) = x_0$

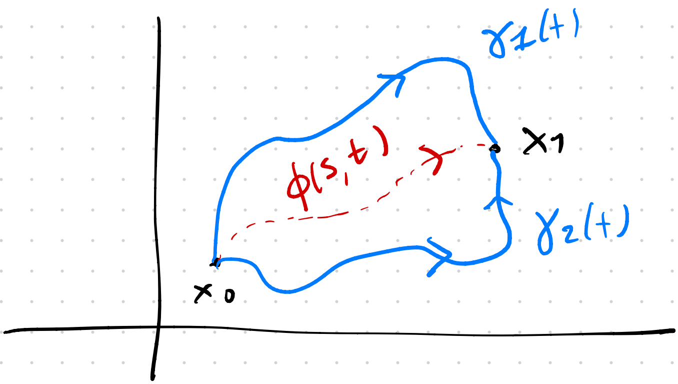
$\gamma_1(1) = \gamma_2(1) = x_1$

are said to be homotopic if there exists a

continuous map $\phi : [0,1] \times [0,1] \to X$ with

$\phi(0,t) = \gamma_1(t)$; $\phi(1,t) = \gamma_2(t)$ $\forall\ 0 \le t \le 1$

$\phi(s,0) = x_0$; $\phi(s,1) = x_1$ $\forall\ 0 \le s \le 1$



we write

$\gamma_1 \sim \gamma_2$

A function $f(\gamma)$ is homotopy invariant if

$$\forall \quad \gamma_1 \sim \gamma_2 \quad f(\gamma_1) = f(\gamma_2)$$

Iterated integrals in general are NOT homotopy invariant

their value depends, instead, on the path chosen!

EXAMPLE 1

$M = \mathbb{R}^2$, and define $\gamma_{r,s} : [0,1] \longrightarrow X$ the

family of paths $\gamma_{r,s}(t) = (t^r, t^s)$ which all

go from $(0,0) \longmapsto (1,1)$ for $\boxed{r, s > 0}$

$\omega_1 = dx \qquad \omega_2 = dy \qquad x, y$ standard coordinates on $\mathbb{R}^2$

$$\int_{\gamma_{r,s}} \omega_1 \omega_2 \quad \int_{0 \le t_1 \le t_2 \le 1} r t_1^{r-1} \, s t_2^{s-1} \, dt_1 \, dt_2$$

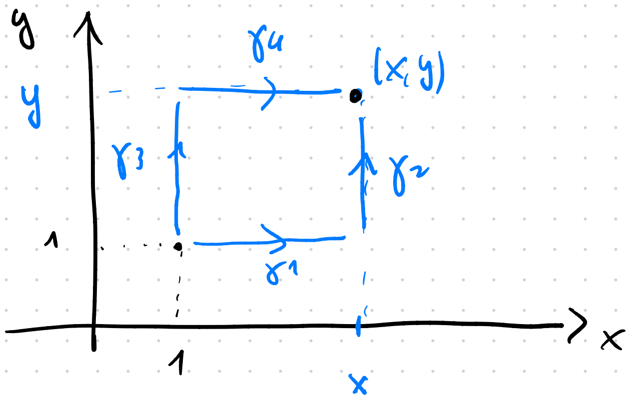
$$= rs \int_0^1 dt_2\, t_2^{s-1} \int_0^{t_2} dt_1\, t_1^{r-1} = s \int_0^1 dt_2\, t_2^{s-1}\, t_2^{r}$$

$$= \frac{s}{r+s} = \int_{\gamma_{\beta\circ S}} \omega_1 \omega_2$$

it depends on the path !

NOT HOMOTOPY INVARIANT

EXAMPLE 2



$$\omega_1 = d\log x$$
$$\omega_2 = d\log y$$

$$\gamma_1(t) = (1 + (x-1)t,\ 1) \qquad \gamma_2(t) = (x,\ 1+(y-1)t)$$

$$\gamma_3(t) = (1,\ 1+(y-1)t) \qquad \gamma_4(t) = (1+(x-1)t,\ y)$$

for $0 \leq t \leq 1$

Use path decomposition formula

$$\int_{\gamma_1\gamma_2} \omega_1\omega_2 = \underbrace{\int_{\gamma_1} \omega_1\omega_2 \int_{\gamma_2}}_{1} + \int_{\gamma_1}\omega_1 \int_{\gamma_2}\omega_2 + \underbrace{\int_{\gamma_1}}_{1} \int_{\gamma_2} \omega_1\omega_2$$

$$\int_{\gamma_1} \omega_1 = \int_0^1 dt \; \frac{d\log(1+(x-1)t)}{dt} = \log(x)$$

$$\int_{\gamma_2} \omega_2 = \int_0^1 dt \; \frac{d\log(1+(y-1)t)}{dt} = \log(y)$$

$$\int_{\gamma_1} \omega_1\omega_2 = \int_{\gamma_2} \omega_1\omega_2 = 0$$

which implies $\displaystyle\int_{\gamma_{12}} \omega_1\omega_2 = \log(x)\log(y)$

$$\int_{\gamma_3\gamma_4} \omega_1\omega_2 = \underbrace{\int_{\gamma_3} \omega_1\omega_2}_{=0} + \underbrace{\int_{\gamma_3}\omega_1 \int_{\gamma_4}\omega_2}_{=0} + \underbrace{\int_{\gamma_4} \omega_1\omega_2}_{=0} = 0!$$

$$\int_{\gamma_{12}} \omega_1\omega_2 = \log x \log y$$

$$\int_{\gamma_{34}} \omega_1\omega_2 = 0$$

result again depends on path!

notice now that if I consider

$$W = \omega_1 ⧢ \omega_2 = \omega_1\omega_2 + \omega_2\omega_1 \quad \text{then}$$

$$\int_{\gamma_{12}} W = \int_{\gamma_{34}} W = \log x \log y$$

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when discussing differential Eqs for Feynman integrals we claimed that the integrability condition on the differential equations guarantees that the resulting functions must be <u>homotopy invariant</u>

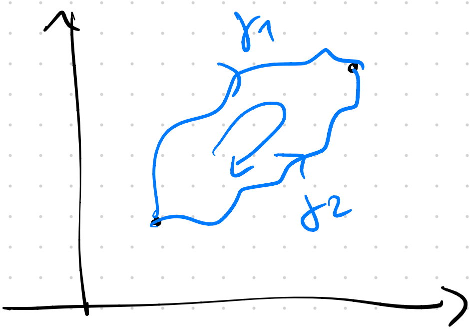
⇒ this means that they cannot generate <u>any</u> iterated integrals, but specific linear combinations!

Let's state precisely the criterion for homotopy invariance of iterated integrals —

. length 1, 1 one form ω_1

γ_1, γ_2 homotopic paths

$$\int_{\gamma_1} \omega_1 = \int_{\gamma_2} \omega_1 \implies \int_{\gamma_1\gamma_2^{-1}} \omega_1 = 0$$



$\gamma_1\gamma_2^{-1}$ = closed curve

if D surface such that $\partial B = \gamma_1\gamma_2^{-1}$ (boundary)

STOKES' THEOREM $$\int_{\gamma_1\gamma_2^{-1}} \omega_1 = \int_{\partial D} \omega_1 = \int_D d\omega_1$$

so $\int_{\gamma_1\gamma_2^{-1}} \omega_1 = 0 \implies \int_D d\omega_1 = 0 \quad \forall \gamma_1, \gamma_2$

$\implies \forall D$!

so this implies $d\omega_1 = 0$, closed differential form

if the form is closed, the iterated integral of length 1 is homotopy invariant !

• Higher length

w = linear comb of words = $a\,\omega_1\omega_2 + b\,\omega_3\omega_4 + \dots$

Homotopy Inv becomes : $[Dw = 0]$ where

$$D(\omega_1 \dots \omega_n) = \sum_{i=1}^{n} \omega_1 \dots d\omega_i \dots \omega_n$$

$$+ \sum_{i=1}^{n-1} \omega_1 \dots (\omega_i \wedge \omega_{i+1}) \dots \omega_n$$

this is nothing but the integrability condition that we already saw for differential equations

$$d\vec{M} = A\vec{M}$$

$$\Rightarrow \boxed{dA - A\wedge A = 0}$$

$\omega_1 \ldots d\omega_i \ldots \omega_n$

A contains differential forms!

note that writing $d\vec{M} + \tilde{A}\vec{M} = 0$ $\quad \tilde{A} = -A$

$$d\tilde{A} + \tilde{A}\wedge\tilde{A} = 0$$ "right" sign!

If ω_i are closed (as for dlogs!) then

$$D(\omega_1 \ldots \omega_n) = \sum_{i=1}^{n-1} \omega_1 \ldots (\omega_i \wedge \omega_{i+1}) \ldots \omega_n$$

this is also the case for iterated integrals that arise from CANONICAL DIFFERENTIAL EQUATIONS, where the integrability condition requires $dB = 0 \Rightarrow$ closed forms

SPECIAL CASE

1 dimensional problem $[\dim X = 1]$

and all differential forms ω_i are holomorphic

$$\Rightarrow \quad \omega = f_z \, dz \qquad \begin{cases} z = x + iy \\ \bar{z} = x - iy \end{cases} \text{ two variables}$$

with $f(z)$ holomorphic function

$$d\omega = \cancel{\frac{\partial f_z}{\partial z} \underbrace{dz \wedge dz}_{=0}} + \underbrace{\frac{\partial f_z}{\partial \bar{z}}}_{=0} d\bar{z} \wedge dz$$

so every 1 dim holomorphic form is closed !

In real 1-dimensional space X

$\omega = g_x \, dx$ $\qquad d\omega = 0$ (there are no two forms!)

more over

$\omega_i \wedge \omega_{i+1} \equiv 0$! because $\underline{dx \wedge dx = 0}$

so if X is one dimensional real space, (or
1 dimensional complex space and all ω_i holomorphic)

then AUTOMATICALLY all words are integrable
and all iterated integrals are homotopy invariant

$\Longrightarrow$ on one-dimensional space there
is "only one path" !

REGULARIZATION

How do we define iterated integrals if an end-point of the path is a singular point?

typical case $\omega = \dfrac{d\xi}{\xi}$ logarithmic form

$$\int_\gamma \omega \rightarrow \int_0^x \frac{d\xi}{\xi} \sim \log(0) = -\infty \; !$$

We want to define a regularised version, which removes these divergences, but preserves properties of iterated integrals.

$\Downarrow$

agree with standard definition when integral CONVERGES

$\Downarrow$

fulfils

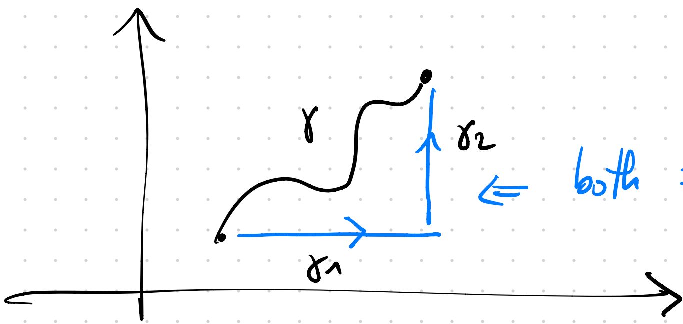
- LINEARITY
- SHUFFLE
- PATH COMPOSITION

$\Rightarrow$ procedure called SHUFFLE REGULARIZATION

(or TANGENTIAL BASE-POINT REG.)

- Assume that all divergences are logarithmic (it can be generalised, but physically well motivated)

- Assume space X is one-dimensional
  ⇒ if it is not, split γ into piecewise constant paths and do the same for each independent "piece"



⇐ both 1-dimensional!

⇒ We ASSUME therefore

$$\begin{cases} \gamma : [0,1] \to [0, x] \\ \omega_i = a_i \, d\log \xi + \dots \end{cases}$$

logarithmic divergence at origin $\xi = 0$

Assume NO other singularity!

Consider

$$\int_\gamma \omega_1 \dots \omega_n = \int_0^x \omega_1 \dots \omega_n$$

define regularized version as follows

1 - Introduce cut off ε, define

$$\int_\gamma \omega_1 \dots \omega_n \equiv \int_{\underline{\underline{\varepsilon}}}^x \omega_1 \dots \omega_n$$

always convergent for $\varepsilon \neq 0$

2 - in limit $\varepsilon \to 0$ we have logarithmic divergences

$$\lim_{\varepsilon \to 0} \int_\varepsilon^x \omega_1 \dots \omega_n = \sum_{k=0}^n I_k(x) \log^k(\varepsilon) + O(\varepsilon)$$

3 - DEFINE $\displaystyle \int_\gamma^{[reg]} \omega_1 \dots \omega_n \equiv \underline{\underline{I_0(x)}}$

throw away all logs!

of course if integral is convergent, definition does not change

$$\int_0^x \omega_1 \dots \omega_n = \lim_{\varepsilon \to 0} \int_\varepsilon^x \omega_1 \dots \omega_n = I_0(x)$$

IMPORTANT

Since for $\varepsilon \neq 0$ integral is convergent, all properties of iterated integrals apply in the same way!

After manipulations, projection onto $I_0(x)$ is consistent with multiplication $\Rightarrow$ regularisation and multiplication COMMUTE.

EXAMPLE

$$\int_0^x \frac{d\xi}{\xi} \to \int_\varepsilon^x \frac{d\xi}{\xi} = \log(x) - \log(\varepsilon) \xrightarrow{\text{reg}} \log(x)$$

note $\log(x) = \int_1^x \frac{d\xi}{\xi}$ change base point!

of course regularised version depends on "scheme choice", I could use

$$\int_{v\varepsilon}^{x} \frac{d\xi}{\xi} = \log(x) - \log(v) - \cancel{\log(\varepsilon)}$$

$$\rightarrow \log(x) - \log(v)$$

it is customary to choose $v=1$ for simplicity

IMPORTANT CASE

if there is only 1 one-form that diverges at $\xi = 0$ we can use use shuffle product to "unshuffle" all this one form and regularization of iterated integrals is reduced to regularizing length 1 integral !

for example $\omega_1 = d\log \xi$ $\quad \omega_2 = d\log(1-\xi)$

$$\int_0^x \omega_1 \omega_2 = \int_0^x \frac{dt}{1-t} \underbrace{\int_0^t \frac{du}{u}}_{\text{divergent !}}$$

use shuffle to write

$$= \underbrace{\int_0^x \omega_1}_{\log(x) - \cancel{\log(\varepsilon)}} \int_0^x \omega_2 - \underbrace{\int_0^x \omega_2 \omega_1}_{\text{now finite}}$$

$$= \log(x) \log(1-x) - \underbrace{\int_0^x \frac{dt}{t} \int_0^t \frac{du}{1-u}}$$

$$= \int_0^x dt \, \frac{\log(1-t)}{t} = -\mathrm{Li}_2(x) \, !$$

$$= \log x \log(1-x) + \mathrm{Li}_2(x)$$ regulated version

SEE NEXT LECTURE FOR DISCUSSION OF POLYLOGS!

LINEAR INDEPENDENCE of iterated integrals

One of the most important properties of iterated integrals in the context of FEYNMAN INTEGRALS is that, under the right assumptions, they are linearly independent

$\Downarrow$

if we express analytical result for a scattering amplitude in terms of iterated integrals, their independence ensures that there are no hidden zeros!

Go back to system of DEQs

$$d\vec{m} = \varepsilon B(x)\, \vec{m}$$

differential forms w_i

$$B = \sum_{i=1}^{n} B_i\, w_i$$

In general, iterated ints $\int_\gamma w_1 \dots w_n$ will NOT be independent:

- Every relation among ω_i induces relations among iterated integrals

- if $\omega_i = df$, then $\int_\gamma \omega_i \to f$ giving new relations

Define ω_i to be linearly DEPENDENT over $\mathcal{C}$ where $\mathcal{C}$ is an algebra of functions

if $\exists$ a function f in $\mathcal{C}$ and $a_i \in \mathbb{C}$ s.t.

$$\sum_i a_i \omega_i = df$$

if $\mathcal{C}$ = anything this is always possible locally !

$\Rightarrow$ we want to restrict $\mathcal{C}$ to be something simple, for example for dlog-forms $\mathcal{C}$ will be rational functions or algebraic functions

Examples (more in next lecture)

- $\omega_1 = d\log(x)$ $\quad$ $\omega_2 = d\log\left(\frac{1}{x}\right)$

clearly $\omega_1 + \omega_2 = 0$

$\Uparrow$ f = constant

- $\omega_1 = d\log\left(\frac{\sqrt{x}+\sqrt{x-4}}{\sqrt{x}-\sqrt{x-4}}\right)$

$\omega_2 = d\log\left(\frac{2+x+\sqrt{x}\sqrt{x+4}}{x}\right)$

$\omega_3 = d\log(x)$

$$\boxed{\omega_1 - \omega_2 - \omega_3 = 0}$$

again f = constant

the iterated integrals arising from $d\bar{m} = \varepsilon B \, \bar{m}$ are linearly independent over $\mathbb{C}$ (as functions) if and only if the one-forms ω_i are linearly independent over $\mathbb{C}$

$\Rightarrow$ we are not proving this in general but we will see examples of this in next lecture when we specialize discussion to the most important class of iterated integrals $\Rightarrow$ MULTIPLE POLYLOGARITHMS