8- Canonical Bases

SS 2022

TUM

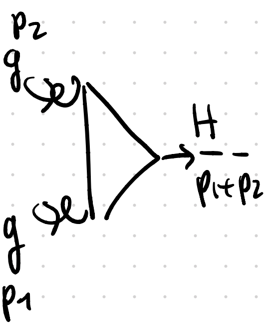
[illegible]

Touhed

In previous lecture we have seen that master integrals fulfil systems of differential equations with rational coefficients in Mandelstam invariants and in the dimensional regulator parameter D

$$d\vec{M} = \underbrace{A(s_{ij}, D)}_{\substack{N\times N \\ \text{matrix}}} \underbrace{\vec{M}}_{\substack{N\text{-vector with} \\ \text{masters}}}$$

In EX 3 derived this for triangle family



$$= \int \frac{d^D k}{\pi^{D/2}} \frac{1}{(k^2-m^2)((k-p_1)^2-m^2)((k-p_{12})^2-m^2)}$$

three masters

$$\{ I(1,0,0),\ I(1,0,1),\ I(1,1,1) \} = M_j$$

Crucial Point

No-one forces us to choose those master integrals! M_j are a basis, as long as we choose the "right number" of MIs, we are free to swap one for the other, for example using IBP for the tadpole

$$I(n+1,0,0) = -\frac{(D-2n)}{2nm^2} I(n,0,0)$$

I could decide to use $I(2,0,0) = -\frac{D-2}{2m^2} I(1,0,1)$

as master integral, instead of $I(1,0,0)$. It turns out that "changing basis" appropriately, form of differential equations can be simplified substantially!

$$d\bar{M} = A\bar{M}$$ change basis as

$$\vec{m} = J\vec{M}$$

$\uparrow$ $N \times N$ matrix (not necessarily rational!)

$$d\vec{m} = dJ\,\bar{M} + J\,d\bar{M} = dJ \cdot J^{-1}\vec{m} + J A J^{-1}\vec{m}$$

new basis satisfies

$$d\vec{m} = [\underbrace{J A J^{-1} + dJ\,J^{-1}}_{B(s_{ij},\, D)}]\,\vec{m}$$

Suppose we can find a matrix $J(s_{ij}, D)$ such that

$$B(s_{ij}, D) = \underbrace{\left(\frac{4-D}{2}\right)}_{\text{"}\epsilon\text{"}}\;\underbrace{\tilde{B}(s_{ij})}_{\text{matrix independent of "D"}}$$

then differential Equations take the suggestive form

$$d\vec{m} = \varepsilon B(s_{ij})\,\vec{m} \qquad (*)$$

ε- FACTORISED DIFFERENTIAL EQS.

Note that finding matrix J in general is as difficult as solving the system of differential Equations for $D \sim 4$

We'll see that, for Feynman Integrals, the matrix J can actually be built in many cases looking at the generalised unitarity cuts of the integrals!

We'll get back to this later — for now let's assume that Eq. (*) has been obtained

why is it a powerful statement?

$\Rightarrow$ We are actually interested in computing master integrals as LAURENT SERIES in ε

Let's expand left and right hand side of (*)

$$\vec{m}_j = \sum_{k=-n_{min}}^{\infty} \varepsilon^k \, \vec{m}_j^{(k)}$$

rescale master integrals such that $\underline{\underline{n_{min} = 0}}$

so we get, at every order in ε:

$$d\vec{m}_j^{(n)} = B(s_{ij}) \, \vec{m}_j^{(n-1)}$$

and explicitly in a given variable x

$$\frac{\partial \vec{m}_j^{(n)}}{\partial x} = B_x(x) \, \vec{m}_j^{(n-1)}(x)$$

chained system, order $n-1$ feeds in into order n

of every order solution is

$$\vec{m}_j^{(n)}(x) = \underbrace{\int^x dt \; B_x(t) \; \vec{m}_j^{(n-1)}(t) + \vec{m}_j^{(n-1)}(x_0)}_{\text{iterated integral over the same "kernels" determined by matrices } \underline{\underline{B_x(t)}}}$$

$$\int^x dt \; B_x(t) \int^t du \; B_x(u) \int^u dv \; B_x(v) \; m_j^{(n-3)}(v)$$

etc

till iteration ends at $\underline{\underline{m_j^{(0)}(x)}}$

Differential Equations naturally expose structure of Feynman Integrals as iterated Integrals over fixed kernels given by matrices $\underline{\underline{B_x}}$!

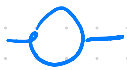
EXAMPLE (1 loop bubble in $D = 2-2\varepsilon$)

Consider again $I(a,b) = e^{\varepsilon\gamma}\int \frac{d^D k}{\pi^{D/2}} \frac{1}{(k^2+m^2)^a\left((k-p)^2+m^2\right)^b}$

γ = Euler Gamma $\cong 0.577216\ldots$

We have seen that there are two master integrals

$M_1 = I(1,0) =$ 

$M_2 = I(1,1) =$ 

$\Rightarrow \quad \vec{M} = \begin{pmatrix} I(1,0) \\ I(1,1) \end{pmatrix}$

Differential Eqs in p^2 where

trivial

$$\frac{d}{dp^2}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ -\frac{(D-2)}{p^2(p^2+4m^2)} & \frac{1}{2}\left(\frac{D-3}{p^2+4m^2} - \frac{1}{p^2}\right) \end{bmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

$$M_1 = I(1,0) = 4\,\Gamma\left(3-\frac{D}{2}\right)\frac{(m^2)^{\frac{D-2}{2}}}{(D-2)(D-4)}\, e^{\left(\frac{2-D}{2}\right)\gamma}$$

Expanding in $(D-2)$

$$M_1 = \frac{2}{2-D} + \frac{2-D}{2}\left(\frac{\pi^2}{12}\right) - \left(\frac{2-D}{2}\right)^2\left(\frac{\zeta_3}{3}\right) + O((D-2)^3)$$

Let us perform the following change of basis

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \frac{2-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p^2(p^2+4m^2)} \end{pmatrix}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

reabsorbs poles so that both integrals become finite in $D\sim 2$

In previous notation

$$\vec{m} = J\,\vec{M}$$

then

$$\frac{d\vec{m}}{dp^2} = \tilde{B}\,\vec{m} \qquad \tilde{B} = JAJ^{-1} + \frac{dJ}{dp^2}J^{-1} \Rightarrow$$

We find from direct calculation

$$\frac{d\vec{m}}{dp^2} = \left(\frac{2-D}{2}\right)\begin{bmatrix} 0 & 0 \\ \dfrac{2}{\sqrt{p^2(p^2+4m^2)}} & \dfrac{1}{p^2+4m^2} \end{bmatrix} \vec{m}$$

$$= \varepsilon \, B(p^2, m^2) \, \vec{m} \qquad \varepsilon = \frac{2-D}{2}$$

We achieved wanted factorization

$\boxed{\text{PRICE TO PAY}}$ $\Rightarrow$ we have now Algebraic functions in B!
not anymore just a rational function!

these equations have another nice feature

the entries of the matrix B are derivatives of logarithms!

$$\frac{2}{\sqrt{p^2(p^2+4m^2)}} \Rightarrow \frac{d}{dp^2}\left[2\log\left(\underbrace{\frac{\sqrt{p^2+4m^2}+\sqrt{p^2}}{\sqrt{p^2+4m^2}-\sqrt{p^2}}}_{f_1(p^2,m^2)}\right)\right]$$

$$\frac{1}{p^2+4m^2} \Rightarrow \frac{d}{dp^2}\left[\ln(\underbrace{p^2+4m^2}_{f_2(p^2,m^2)})\right]$$

in differential form equations become

$$d\vec{m} = \varepsilon \begin{bmatrix} 0 & 0 \\ \omega_1 & \omega_2 \end{bmatrix} \vec{m}$$

$$\omega_1 = d\log(f_1(p^2,m^2))$$

$$\omega_2 = d\log(f_2(p^2,m^2))$$

these are called d-log forms

result is very nice => we expect scattering amplitudes to have LOGARITHMIC SINGULARITIES from causality + locality + unitarity !

these "special" Feynman Integrals fulfil DEQs which make MANIFEST that they can only have logarithmic singularities

=> they have property to make them good BUILDING BLOCKS for physical scattering amplitudes

they deserve a name of their own

CANONICAL MASTER INTEGRALS

$\vec{m}$ => CANONICAL BASIS in $\underline{D \sim 2}$

(notice in $D=4$ the basis would be different !)

To SUMMARISE

any even number
DIM SHIFT !!

If we can find a change of basis J such that differential Eqs become (in $D = n-2\varepsilon$)

$$d\vec{m} = \varepsilon\, B(s_{ij}, m^2) \cdot \vec{m}$$

with B in d-log form

then the basis $\vec{m}$ is called CANONICAL

J. Henn '13
arXiv 1304.1806

NOTE THAT we know that there exist examples when such basis cannot be found! Differential forms in matrix B could go beyond logarithms!

ACTIVE RESEARCH TOPIC !

FORMAL PROPERTIES OF CANONICAL INTEGRALS

Let's go back to our canonical basis

$$d\vec{m} = \varepsilon B(x)\, \vec{m}$$

where x is any of the s_{ij} or m_j^2 (masses)

Assume also that we also know boundary $\vec{m}(x_0,\varepsilon) = \vec{m}_0(\varepsilon)$

Solution can be obtained formally as

$$\vec{m}(x,\varepsilon) = \mathbb{P}\exp\left[\varepsilon \int_\gamma B(x')\right] \vec{m}_0(\varepsilon)$$

γ is a path from x_0 to x

$$W(\gamma,\varepsilon) = \mathbb{P}\exp\left[\varepsilon \int_\gamma B(x')\right]$$

PATH ORDERED EXPONENTIAL

Since equations are in canonical form, $W(\gamma,\varepsilon)$ can be obtained very easily as expansion in ε

For example if

one-forms

$$B(x) = B_1 \omega_1 + B_2 \omega_2$$

Constant matrices (numbers!)

$$W(\gamma,\varepsilon) = \mathbb{1} + \varepsilon\left(B_1 \int_\gamma \omega_1 + B_2 \int_\gamma \omega_2 \right)$$

$$+ \varepsilon^2 \left[B_1 B_1 \int_\gamma \omega_1 \omega_1 + B_1 B_2 \int_\gamma \omega_2 \omega_1 + \right.$$

$$\left. + B_2 B_1 \int_\gamma \omega_1 \omega_2 + B_2 B_2 \int_\gamma \omega_2 \omega_2 \right]$$

$$+ O(\varepsilon^3)$$

iterated integrals

see next lecture !!

- If basis is CANONICAL the iterated integrals can only have logarithmic singularities

In particular if $f(x) \to 0$ then we expect

$$\int_\gamma \omega_1 \ldots \omega_n \sim \sum_{k=0}^{n} c_k \ln^k(f(x)) + \ldots$$

constant numbers

- Result will be linear combination of the iterated integrals $\int_\gamma \omega_1 \ldots \omega_j$ with NUMERICAL COEFFICIENTS [no rational, algebraic, etc function]

Functions with this behaviour are called

PURE FUNCTIONS $\Rightarrow$ stronger constraint than "just any" iterated integrals!

• INTEGRABILITY CONDITION

for standard basis was

$$\boxed{dA - A \wedge A = 0}$$

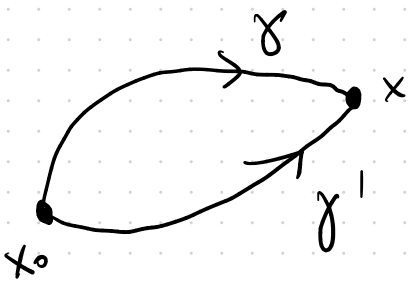
now becomes $\left(d^2 m = 0 ! \right)$

$$\varepsilon \, dB(x) - \varepsilon^2 B(x) \wedge B(x) = 0$$

$$\Rightarrow \begin{cases} dB = 0 & \Rightarrow \text{matrix of closed forms!} \\ B \wedge B = 0 \end{cases}$$

first condition obvious for d-logs, since dlogarithms are by construction closed FORMS

⇒ Important, both for standard (!) and canonical form, INTEGRABILITY CONDITION guarantees that path ordered exponential does not depend on choice of path γ AS LONG AS γ, γ' can be continuously deformed in each other!



difference is a closed path
if FORMS EXACT difference must be zero...

more on this in the next lecture

HOW TO FIND CANONICAL BASIS ?

Look at standard Eq. for Bubble

$$\frac{d}{dp^2}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ -\frac{(D-2)}{p^2(p^2+4m^2)} & \frac{1}{2}\left(\frac{D-3}{p^2+4m^2} - \frac{1}{p^2}\right) \end{bmatrix}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

⇑ not in ε-factorised form

in limit $D=2$ we get

$$\frac{d}{dp^2}\begin{pmatrix} M_1 \\ \boxed{M_2} \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{2}\left(\frac{1}{p^2} + \frac{1}{p^2+4m^2}\right) \end{bmatrix}\begin{pmatrix} M_1 \\ \boxed{M_2} \end{pmatrix}$$

"offending" term

M_2 has non trivial homogeneous solution

$$\frac{d}{dp^2} M_2^H = -\frac{1}{2}\left(\frac{1}{p^2} + \frac{1}{p^2+4m^2}\right) M_2^H$$

$$M_2^H(D=2) = \frac{C}{\sqrt{p^2(p^2+4m^2)}}$$

up to overall prefactor $\frac{2-D}{2}$, new bonds defined exactly with factor that makes it homogeneous equation trivial

$$m_2 = \sqrt{p^2(p^2+4m^2)}\, M_2 \Rightarrow m_2^H(D=2) = C$$

just a constant

in fact its equation is

$$\frac{d}{dp^2} m_2^H = \left(\frac{2-D}{2}\right)\frac{1}{p^2+4m^2} m_2^H$$

which in $D=2$ is just

$$\frac{d}{dp^2} m_2^H = 0 \Rightarrow \underline{m_2^H = C}$$

FIRST CLUE : J is build from homogeneous solutions in $D=2$ (or $D=4, 2n$ etc...)

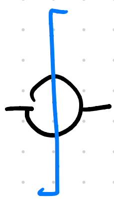
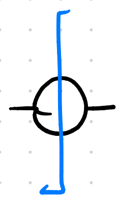
depending what we are interested in

How to Find Homogeneous Solutions ?

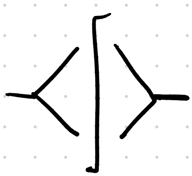
Generalised Unitarity cuts satisfy similar differential Equations as the UNCUT integrals !

$\Rightarrow$ MAXIMAL CUT satisfies homogeneous Equation

see [1610.08397, 1704.05465]

$\frac{d}{dp^2}$  $= c_1(p^2, m^2, D)$  $+ \cancel{c_2(p^2, m^2, D)}$ 

no support on the cut !

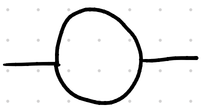
In fact  $\propto \frac{1}{\sqrt{p^2(p^2+4m^2)}}$ in $D=2$

easy to prove with $\underline{\text{BAIKOV REPR.}}$ [1701.07356]

MAX CUT is only first part of the story –

Canonical differential Equations mean that Integrals can be written as iterated integrals over dlogs

$\Rightarrow$ We can "guess" if this is going to be the case, just looking at the INTEGRAND, ie without even deriving differential equations!



$$= \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2+m^2)((k-p)^2+m^2)}$$

Close to $D=2$ $\quad k = \alpha p_1 + \beta p_2 \quad p = p_1 + p_2$

$p_i^2 = 0 \quad i = \{1,2\}$

$$k^2 = 2\alpha\beta\, p_1 \cdot p_2 = \alpha\beta\, p^2$$

$$(k-p)^2 = (\alpha-1)(\beta-1)\, p^2$$

$$d^2k = \frac{p^2}{2}\, d\alpha\, d\beta$$ $\Rightarrow$ prove it!

Integral becomes (in $D=2$!)

$$\frac{p^2}{2}\int d\alpha\, d\beta \frac{1}{\left(p^2\alpha\beta+m^2\right)\left(p^2(\alpha-1)(\beta-1)+m^2\right)} =$$

$$= \frac{1}{2p^2}\int_{\epsilon} d\alpha\, d\beta \frac{1}{\left(\alpha\beta+\frac{m^2}{p^2}\right)\left(\underbrace{(\alpha-1)(\beta-1)}_{\alpha\beta-\alpha-\beta+1}+\frac{m^2}{p^2}\right)}$$

don't care about ϵ now

$$= \frac{1}{2p^2}\int d\beta \int d\alpha \left[\frac{\beta}{\left(\alpha\beta+\frac{m^2}{p^2}\right)} - \frac{(\beta-1)}{(\alpha-1)(\beta-1)+\frac{m^2}{p^2}}\right]\frac{1}{\frac{m^2}{p^2}+\beta(1-\beta)}$$

$$= \frac{1}{2p^2}\int \frac{d\beta}{\beta(1-\beta)+\frac{m^2}{p^2}}\left[\int \frac{d\log\left(\alpha\beta+\frac{m^2}{p^2}\right)}{d\alpha}d\alpha - \int \frac{d\log\left[(\alpha-1)(\beta-1)+\frac{m^2}{p^2}\right]}{d\alpha}d\alpha\right]$$

also a dlog!

dlogs!

Indeed

$$\int \frac{d\beta}{\frac{m^2}{p^2}+\beta(1-\beta)} = -\int \frac{d\beta}{\left(\beta-\left(\frac{1}{2}+\frac{\sqrt{1+\frac{4m^2}{p^2}}}{2}\right)\right)\left(\beta-\left(\frac{1}{2}-\frac{\sqrt{}}{2}\right)\right)}$$

$$= + \frac{1}{\sqrt{1+\frac{4m^2}{p^2}}}\left\{\int \frac{d\beta}{\beta-\left(\frac{1}{2}+\frac{\sqrt{1+\frac{4m^2}{p^2}}}{2}\right)} - \int \frac{d\beta}{\beta-\left(\frac{1}{2}-\frac{\sqrt{1+\frac{4m^2}{p^2}}}{2}\right)}\right\}$$

$$= \frac{\sqrt{p^2}}{\sqrt{p^2+4m^2}}\left\{\int \frac{d\log\left(\beta-\left(\frac{1}{2}+\frac{\sqrt{}}{2}\right)\right)}{d\beta}\,d\beta - \int \frac{d\log(\;)}{d\beta}\,d\beta\right\}$$

so we discovered that in $D=2$

→ exactly what we expect from CANONICAL INT

$$-\bigcirc- = \frac{1}{2p^2}\frac{\sqrt{p^2}}{\sqrt{p^2+4m^2}}\left\{\int d\log \int d\log + \int d\log \int d\log + \ldots\right\}$$

$$\parallel$$

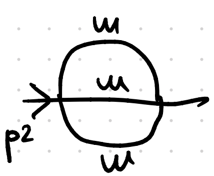
$$\frac{1}{\sqrt{p^2(p^2+4m^2)}}$$ exactly solution of hom eq = max cut !

In general, the maximal cut might be more complicated than just a rational or algebraic function $\Rightarrow$ STILL, **CONJECTURALLY**, it is always possible to construct matrix J to get differential equations in ε-factorised form. The matrix will in general **NOT** be in d-log form, but it will contain transcendal functions that originate from **MAXIMAL CUT**

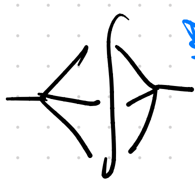
The case of d-log forms plays a particularly important role in Feynman Integral calculation

Extending this to more general case is current research topic —

EXAMPLE



$\Rightarrow$



$$\overset{D=2}{=} \frac{1}{\sqrt{P(p^2/m^2)}} \, \overbrace{K\left(Q(p^2/m^2)\right)}^{\text{Elliptic Integral of FIRST KIND}}$$