7 - Differential Equations Method

2/2

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TUM

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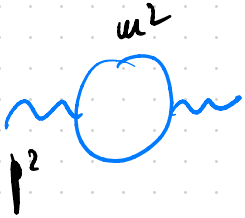
In previous lecture we have seen a very simple example of the differential equation method to a problem that depends only on two scales = 1 dimensionless ratio



$$= f\left(\frac{p^2}{m^2}\right) \cdot (m^2)^{\#}$$

dimensional analysis fixes this.

Let's generalise this construction:

1] Feynman Integrals are homogeneous functions of the external invariants and internal masses

$$= f\Big(\underbrace{s_{12}, s_{13}, \dots, s_{1N}, \dots}_{s_{ij}}, \underbrace{m_1^2, \dots m_M^2}_{m_j^2}\Big)$$

by dimensional analysis we have

$$f(\lambda^2 s_{ij}, \lambda^2 m_j^2) = \lambda^{+\alpha} f(s_{ij}, m_j^2)$$

so λ has dimension of $[m]$

homogeneous of degree $+\alpha$

2] Feynman integrals fulfil Integration-by-parts identities

$$\int \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \left(\frac{S_1^{a_1} \cdots S_\sigma^{a_\sigma}}{D_1^{b_1} \cdots D_\tau^{b_\tau}} \right) = I(b_1, \dots, b_\tau, -a_1, \dots, -a_\sigma)$$

the integral does not change under transformation

$$k_i^\mu \to A_{ij} k_j^\mu + B_{ij} p_j^\mu = M_{ij} v_j^\mu$$

$\det A \neq 0$

external momenta

$v_j^\mu = (k_i^\mu, p_j^\mu)$

Form a Lie group in space of Feyn. Ints!

Write integral as

$$\int \prod_{\ell=1}^{L} \frac{d^D k_\ell}{(2\pi)^D} f(v_j)$$

Let's specialize this to infinitesimal form

$$k_i \to k_i + \alpha v_j \quad \begin{cases} j = i \\ j \neq i \end{cases} \quad \text{both included}$$

then

$$f \to f + \alpha v_j \cdot \partial_i f$$

and for $j = i$ we have

$$d^D k_i \to (1 + \alpha D)\, d^D k_i$$

Imposing that the integral does not change we have

$$\int \prod_{\ell=1}^{L} \frac{d^D k_\ell}{(2\pi)^D} \left[O_{ij} f \right] = 0$$

$$O_{ij} = \partial_i \cdot v_j = \frac{\partial}{\partial q_i} v_j \qquad \left\{ \begin{array}{l} i \leq L \\ j \geq i \end{array} \right\}$$

derivatives wrt loop momenta

these are the IBPs that we previously wrote as

$$\int \prod_{\ell=1}^{L} \frac{d^D k_\ell}{(2\pi)^D} \frac{\partial}{\partial k_i^\mu} \left[v_j^\mu \frac{S_1^{a_1} \cdots S_\sigma^{a_\sigma}}{D_1^{b_1} \cdots D_\tau^{b_\tau}} \right] = 0$$

One can study group properties / Lie algebra generated by O_{ij}, we won't do it here.

3] Feynman Integrals fulfil LORENTZ IDENTITIES

Feynman Ints are scalar functions, they should be invariant under a Lorentz transf of external momenta

Infinitesimally: $p^\mu \to p^\mu + \delta p^\mu = p^\mu + \delta \varepsilon^\mu{}_\nu \, p^\nu$

$$\underbrace{\delta \varepsilon^\mu{}_\nu = -\delta \varepsilon^\nu{}_\mu}_{\text{infinitesimal generators Lorentz group}}$$

$I(p_i + \delta p_i) \equiv I(p_i)$ Lorentz Scalar

Expanding

$$I(p_i + \delta p_i) = I(p_i) + \sum_{j=1}^{E} \delta p_j^\mu \frac{\partial}{\partial p_j^\mu} I(p_i)$$

which in turn becomes

$$\delta \varepsilon^{\mu}{}_{\nu} \left[p_1^{\nu} \frac{\partial}{\partial p_1^{\mu}} + \dots + p_E^{\nu} \frac{\partial}{\partial p_E^{\mu}} \right] I(p_i) = 0$$

all external momenta

now $\delta \varepsilon^{\mu}{}_{\nu}$ has 6 components (antisymmetric tensor)

there are <u>up to</u> 6 LIs, in fact using antisymm

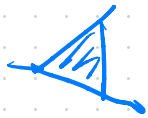
$$\Rightarrow \left[p_1^{\nu} \frac{\partial}{\partial p_1^{\mu}} - p_1^{\mu} \frac{\partial}{\partial p_1^{\nu}} + \dots + p_E^{\nu} \frac{\partial}{\partial p_E^{\mu}} - p_E^{\mu} \frac{\partial}{\partial p_E^{\nu}} \right] I(p_i) = 0 \qquad (*)$$

can be projected to scalar identities, by contracting it with all <u>antisymmetric</u> <u>combinations</u> of $\underline{p_{i\mu}}\,\underline{p_{j\nu}}$

EXAMPLES

- 2-point integrals [ ]

  1 momentum p^μ, zero LIs

- 3-point functions []

  2 momenta, p_1^μ, p_2^μ, 1 LIs $(p_1^\mu p_2^\nu - p_1^\nu p_2^\mu)$

  $\parallel$

  $p_{[1}^\mu, p_{2]}^\nu$

- 4-point functions [ ]

  3 momenta $p_1^\mu, p_2^\mu, p_3^\mu$; 3 LIs $\left\{ \begin{array}{l} p_{[1}^\mu, p_{2]}^\nu \\ p_{[1}^\mu, p_{3]}^\nu \\ p_{[2}^\mu, p_{3]}^\nu \end{array} \right\}$

- 5-point functions [ ]

  4 momenta $p_1^\mu \dots p_4^\mu$, $\underline{\underline{6 \text{ LIs}}}$! $\left\{ \begin{array}{l} p_{[1}^\mu, p_{2]}^\nu, p_{[1}^\mu, p_{3]}^\nu, p_{[1}^\mu, p_{4]}^\nu \\ p_{[2}^\mu, p_{3]}^\nu \quad p_{[2}^\mu, p_{4]}^\nu \\ p_{[3}^\mu p_{4]}^\nu \end{array} \right.$

→

So only starting at 5-point, we can project out out <u>ALL LIs</u>.

In practise, we act with (*) on the <u>INTEGRAND</u> and contract it with all combinations of $p^\mu_{[i}, p^\nu_{j]}$ and in this way obtain new identities between Feynman Integrals

- one can prove that LIs are <u>NOT</u> linear independent from <u>IBPs</u>

- they are nevertheless helpful to find all relations among integrals "<u>more easily</u>"

- they help to understand how to derive differential equations for Feynm. Integrals in general

4] Differential Equations

Scalar Feynman Ints only depend on $s_{ij} = (p_i + p_j)^2$

$= 2 p_i \cdot p_j$

not on the momenta themselves

massless case!

CHAIN RULE

$$\frac{\partial}{\partial p_{i\mu}} = \sum_j 2(p_{i\mu} + p_{j\mu}) \frac{\partial}{\partial s_{ij}}$$

substituting this into general Lorentz Id

$$\underbrace{\left(p_1^\nu \frac{\partial}{\partial p_1^\mu} - p_1^\mu \frac{\partial}{\partial p_1^\nu} + \dots + p_E^\nu \frac{\partial}{\partial p_E^\mu} - p_E^\mu \frac{\partial}{\partial p_E^\nu} \right)}_{} I = 0$$

this bracket is identically zero

the LIs become trivial identities !

BUT remember, non-trivial at the

<u>INTEGRAND</u> <u>Level</u> !

the crucial point is that, if I want to express now $\frac{\partial}{\partial s_{ij}} = f\left(p_i^\mu \frac{\partial}{\partial p_j^\mu}\right)$ there is a redundancy hidden in Lorentz ids! I can always add a combination of LIs to $\frac{\partial}{\partial s_{ij}}$, it will not change final result, since any LIs applied on Feynman Ints gives zero!

EXAMPLES

- 2 point: p^μ ; $p^\mu p_\mu = s \Rightarrow \frac{\partial}{\partial p^\mu} = \frac{\partial s}{\partial p^\mu}\frac{\partial}{\partial s} = 2p_\mu \frac{\partial}{\partial s}$

so $p^\mu \frac{\partial}{\partial p^\mu} = 2s\frac{\partial}{\partial s} \Rightarrow \frac{\partial}{\partial s} = \frac{1}{2s} p^\mu \frac{\partial}{\partial p^\mu}$

zero LIs, no ambiguity!

• 3 point: p_1^μ, p_2^μ 2 momenta

$$\begin{cases} 3\ s_{ij} \Rightarrow p_1^2,\ p_2^2,\ s_{12} = (p_1+p_2)^2 = p_1^2 + p_2^2 + 2p_1\cdot p_2 \\ 1\ LI \Rightarrow (p_{1\mu}p_{2\nu} - p_{1\nu}p_{2\mu})\ (*) \end{cases}$$

Invertible modulo LIs!

= 4 combinations

$$p_i^\mu \frac{\partial}{\partial p_{j\mu}} \qquad i,j = \{1,2\}$$

$$\frac{\partial}{\partial p_{1\mu}} = \frac{\partial p_1^2}{\partial p_{1\mu}} \frac{\partial}{\partial p_1^2} + \frac{\partial s_{12}}{\partial p_{1\mu}} \frac{\partial}{\partial s_{12}}$$

$$= 2p_1^\mu \frac{\partial}{\partial p_1^2} + 2(p_1^\mu + p_2^\mu) \frac{\partial}{\partial s_{12}}$$

$$\boxed{\begin{aligned} p_{1\mu} \frac{\partial}{\partial p_{1\mu}} &= 2p_1^2 \frac{\partial}{\partial p_1^2} + (s_{12} + p_1^2 - p_2^2) \frac{\partial}{\partial s_{12}} \qquad (1) \\ p_{2\mu} \frac{\partial}{\partial p_{1\mu}} &= (s_{12} - p_1^2 - p_2^2) \frac{\partial}{\partial p_1^2} + (s_{12} + p_2^2 - p_1^2) \frac{\partial}{\partial s_{12}} \qquad (2) \end{aligned}}$$

$$\frac{\partial}{\partial p_{2\mu}} = 2 p_{2\mu} \frac{\partial}{\partial p_2^2} + 2(p_1^\mu + p_2^\mu) \frac{\partial}{\partial s_{12}}$$

$$\boxed{\begin{aligned} p_{1\mu} \frac{\partial}{\partial p_{2\mu}} &= (s_{12} - p_1^2 - p_2^2) \frac{\partial}{\partial p_2^2} + (s_{12} + p_1^2 - p_2^2) \frac{\partial}{\partial s_{12}} \qquad (3) \\ p_{2\mu} \frac{\partial}{\partial p_{2\mu}} &= 2 p_2^2 \frac{\partial}{\partial p_2^2} + (s_{12} + p_2^2 - p_1^2) \frac{\partial}{\partial s_{12}} \qquad (4) \end{aligned}}$$

system not invertible for $\left\{ \frac{\partial}{\partial s_{12}}, \frac{\partial}{\partial p_1^2}, \frac{\partial}{\partial p_2^2} \right\}$!

LI reads

$$(p_{1\mu} p_{2\nu} - p_{2\mu} p_{1\nu}) \left(p_1^\mu \frac{\partial}{\partial p_1^\nu} - p_1^\nu \frac{\partial}{\partial p_1^\mu} + p_2^\mu \frac{\partial}{\partial p_2^\nu} - p_2^\nu \frac{\partial}{\partial p_2^\mu} \right) =$$

$$= 2 p_1^2 \, p_{2\mu} \frac{\partial}{\partial p_{1\mu}} - 2 p_1 \cdot p_2 \, p_{1\mu} \frac{\partial}{\partial p_{1\mu}} + 2 p_1 \cdot p_2 \, p_{2\mu} \frac{\partial}{\partial p_{2\mu}} - 2 p_2^2 \, p_{1\mu} \frac{\partial}{\partial p_{2\mu}}$$

LI becomes

$$2p_1^2\, p_{2\mu} \frac{\partial}{\partial p_{1\mu}} - 2p_2^2\, p_{1\mu} \frac{\partial}{\partial p_{2\mu}} +$$

$$+ \underbrace{2\, p_1 \cdot p_2}_{s_{12} - p_1^2 - p_2^2} \left[p_{2\mu} \frac{\partial}{\partial p_{2\mu}} - p_{1\mu} \frac{\partial}{\partial p_{1\mu}} \right] = 0$$

when applied on a Feynman integral (SCALAR!)

One of the $p_{i\mu} \frac{\partial}{\partial p_{j\mu}}$ can always be removed in terms of the others!

and the system becomes invertible, we can express all $\left\{ \frac{\partial}{\partial s_{12}}, \frac{\partial}{\partial p_1^2}, \frac{\partial}{\partial p_2^2} \right\}$ in terms of the $p_{i\mu} \frac{\partial}{\partial p_{j\mu}}$

In particular, I can choose to remove <u>difference</u> and keep only

$$p_{1\mu}\frac{\partial}{\partial p_{2\mu}}\,, \quad p_{2\mu}\frac{\partial}{\partial p_{1\mu}}\,, \quad \left(p_{1\mu}\frac{\partial}{\partial p_{1\mu}} + p_{2\mu}\frac{\partial}{\partial p_{2\mu}}\right)$$

which means using <u>2</u>, <u>3</u>, <u>1 + 4</u>

$$p_{2\mu}\frac{\partial}{\partial p_{1\mu}} = (s_{12}-p_1^2-p_2^2)\frac{\partial}{\partial p_1^2} + (s_{12}+p_2^2-p_1^2)\frac{\partial}{\partial s_{12}} \quad (2)$$

$$p_{1\mu}\frac{\partial}{\partial p_{2\mu}} = (s_{12}-p_1^2-p_2^2)\frac{\partial}{\partial p_2^2} + (s_{12}+p_1^2-p_2^2)\frac{\partial}{\partial s_{12}} \quad (3)$$

$$\left(p_{1\mu}\frac{\partial}{\partial p_{1\mu}} + p_{2\mu}\frac{\partial}{\partial p_{2\mu}}\right) = 2\left(p_1^2\frac{\partial}{\partial p_1^2} + p_2^2\frac{\partial}{\partial p_2^2}\right) + 2s_{12}\frac{\partial}{\partial s_{12}} \quad (1)+(4)$$

if $p_1^2=0$, $p_2^2=0$ recover case of exercise sheet <u>3</u>.

• Le Point, 3 momenta $p_1^\mu, p_2^\mu, p_3^\mu \Longrightarrow$

$p_1^2, p_2^2, p_3^2, s_{12}, s_{13}, s_{23}$ 6 variables in total

$$p_{i\mu} \frac{\partial}{\partial p_{j\mu}} = 9 \text{ combinations}$$

there are indeed 3 LIs $[p_{1\mu} p_{2\nu}]$, $[p_{1\mu} p_{3\nu}]$, $[p_{2\mu} p_{3\nu}]$ etc

[etc for higher point]

EULER SCALING RELATION

Since FeynmanInts are homogeneous functions, the derivatives are not all independent

$$I(\lambda^2 s_{ij}, \lambda^2 m_j^2) = f(\lambda, s_{ij}, m_j^2)$$

$$\parallel$$

$$\lambda^2 I(s_{ij}, m_j^2) \quad \text{homogeneity}$$

Differentiating wrt λ we find:

$$\frac{\partial}{\partial \lambda} I(\lambda^2 s_{ij}, \lambda^2 m_j^2) = \frac{\partial}{\partial \lambda}\left(\lambda^\alpha I(s_{ij}, m_j^2)\right)$$

$$\Longrightarrow$$

$$\sum_{kn} s_{kn} \frac{\partial}{\partial s_{kn}} I(\lambda^2 s_{ij}, \lambda^2 m_j^2) + \sum_k m_k^2 \frac{\partial}{\partial m_k^2} I(\lambda^2 s_{ij}, \lambda^2 m_j^2)$$

$$= \frac{\alpha}{2} \lambda^\alpha I(s_{ij}, m_j^2)$$

For $\lambda = 1$ it gives <u>SCALING RELATION</u>

$$\left[\sum_{kn} s_{kn} \frac{\partial}{\partial s_{kn}} + \sum_k m_k^2 \frac{\partial}{\partial m_k^2}\right] I(s_{ij}, m_j^2) = \frac{\alpha}{2} I(s_{ij}, m_j^2)$$

Dimensional scaling

$$\alpha = LD + 2S - 2R$$

\# Loops

powers scal prod

tot powers of propagators

this reflects the fact that we can always put one scale $= 1$ and only consider dimensionless ratios!

$\lambda^2 = \frac{1}{m_1^2}$ for example removes dependence on one of the <u>masses</u>.

<u>EXAMPLE</u> 1 loop bubble

In previous lecture we derived: $I(a,b) = \int \frac{d^D k}{(k^2+m^2)((k-p)^2+m^2)}$

$$p^2 \frac{\partial}{\partial p^2} \underbrace{I(1,1)}_{B(p^2)} = \frac{p^2}{2p^2}\left[I(0,2) - I(1,1) \right]$$

$$- \frac{p^2}{2} I(1,2)$$

the other derivative is

$$m^2 \frac{\partial}{\partial m^2} I(1,1) = -m^2 I(1,2) - m^2 I(2,1) = -2m^2 I(1,2)$$

so that

$$\left(p^2 \frac{\partial}{\partial p^2} + m^2 \frac{\partial}{\partial m^2}\right) I(1,1) = -\frac{1}{2} I(1,1) - \frac{(p^2+4m^2)}{2} I(1,2) + \frac{1}{2} I(0,2)$$

and using

$$I(0,2) = -\frac{(D-2)}{2m^2} \overbrace{I(1,0)}^{T(m^2)}$$

$$I(1,2) = -\frac{(D-2)}{2m^2(p^2+4m^2)} \overbrace{I(1,0)}^{T(m^2)} - \frac{(D-3)}{p^2+4m^2} \underbrace{I(1,1)}_{B(p^2)}$$

$$= \left(\frac{D-3}{2}\right) I(1,1) - \frac{1}{2} I(1,1) + \cancel{\frac{(D-2)}{4m^2} I(1,0)} - \cancel{\frac{(D-2)}{4m^2} I(1,0)}$$

$$\boxed{\left(p^2 \frac{\partial}{\partial p^2} + m^2 \frac{\partial}{\partial m^2}\right) I(1,1) = \left(\frac{D-4}{2}\right) I(1,1)}$$

SCALING RELATION

CONCLUSION

Given a family of L-loop integrals

- Use IBPs (and Lorentz Ids) to reduce them to N master integrals M_i

- Differentiate wrt all independent s_{ij} and m_j^2

$x_k = (s_{ij}, m_j^2)$ all variables

$$\frac{\partial M_i}{\partial x_k} = A_{ij}^{(x_k)} M_j \Rightarrow \frac{\partial}{\partial x_k} \vec{M} = [A_k] \vec{M}$$

$N \times N$ matrix with rational coefficients in x & D

$\Rightarrow$ obvious because IBPs (and LIs) only produce rational coefficients!

- Useful to recollect all equations in form of total differential (language of differential forms)

$$d\vec{M} = \sum_{k=1}^{N} \frac{\partial \vec{M}}{\partial x_k} dx_k$$

$$A = \sum_{k=1}^{N} A_k \, dx_k$$

matrix-valued one form !

so we can write

$$\boxed{d\vec{M} = A \, \vec{M}} \Rightarrow (d - A)\vec{M} = 0$$

the system of differential Eqs must <u>be integrable</u>

$\Rightarrow$ the result $(\vec{M})$ must be a "proper function"

which means $\frac{\partial M}{\partial x_i \partial x_j} = \frac{\partial M}{\partial x_j \partial x_i} \Leftrightarrow d^2 M = 0$

start from $\frac{\partial \vec{M}}{\partial x_i} = A_i \vec{M}$ & $\frac{\partial \vec{M}}{\partial x_j} = A_j \vec{M}$

$$\frac{\partial^2 \vec{M}}{\partial x_j \partial x_i} = \frac{\partial A_i}{\partial x_j} \vec{M} + A_i \overbrace{\frac{\partial \vec{M}}{\partial x_j}}^{A_j \vec{M}} \qquad (1)$$

$$\frac{\partial^2 \vec{M}}{\partial x_i \partial x_j} = \frac{\partial A_j}{\partial x_i} \vec{M} + A_j \underbrace{\frac{\partial \vec{M}}{\partial x_i}}_{A_i \vec{M}} \qquad (2)$$

- <u>(1)-(2) gives</u>

$$\left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} \right) \vec{M} + \left(A_i A_j - A_j A_i \right) \vec{M} = 0$$

$$\underbrace{\left[\left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} \right) + [A_i, A_j] \right]}_{\text{INTEGRABILITY CONDITION ON MATRICES } \underline{A_i}\,!} \vec{M} = 0$$

Can also be rewritten in language of DIFFERENTIAL FORMS

A one form $\quad A = A_i \, dx_i$

then its <u>total differential</u> = <u>Exterior derivative</u>

$$dA = \frac{\partial A}{\partial x_j \partial x_i} \underbrace{dx_j \wedge dx_i}_{\text{antisymmetric}}$$

antisymmetrize:

$$= \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} \right) dx_j \wedge dx_i$$

Similarly

$$A \wedge A = A_i A_j \underbrace{dx_i \wedge dx_j}_{\text{antisymmetric}}$$

$$= [A_i, A_j] \, dx_i \wedge dx_j$$

So in diff. forms language INTEGRABILITY

$$\boxed{dA - A \wedge A = 0}$$