6- Differential Eqs Method

SS 2022

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Until now we have spent some time discussing how to Decompose scattering Amplitudes in terms of "master integrals".

We left the issue of the rational part open for now, there are two solutions

Unitarity in higher dimensions or with masses

INTEGRATION BY PARTS IDENTITIES <u>IBPs</u>

IBPs were shortly discussed in previous course. We will re-discuss them here and go further, showing how they can be used to derive DIFFERENTIAL EQUATIONS satisfied by the master integrals, which can be used to substantially simplify their analytic or numerical calculation!

(RECAP ON) INTEGRATION BY PARTS

these are identities among integrals of a given "type", and are the most general way to perform a reduction to master integrals, from the mathematical (not "bound" to scattering amplitude") point of view!

the idea is GAUSS THEOREM in d-dimensions

Let's give some definitions, and then state it

INTEGRAL FAMILY

$$\int \prod_{l=1}^{L} \frac{d^D k_l}{\pi^{D/2}} \; \frac{\overbrace{S_1^{a_1} \cdots S_\sigma^{a_\sigma}}^{\text{set of irreducible scalar products (ISPs)}}}{\underbrace{D_1^{b_1} \cdots D_\tau^{b_\tau}}_{\text{set of propagators}}} = I(\underbrace{b_1, \dots, b_\tau, -a_1, \dots, -a_\sigma}_{\text{integers} \in \mathbb{Z}})$$

because starting @ 2loops, not all scalar products can be written as propagators!

At 1 loop the $D_i^{b_i}$ are enough!

IBPs

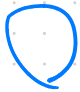
any vector $\left\{ \begin{matrix} k^\mu \\ p_j^\mu \end{matrix} \right\}$

$$\int \prod_{l=1}^{L} \frac{d^D k_l}{\pi^{D/2}} \; \frac{\partial}{\partial k_j^\mu} \left[v^\mu \, \frac{S_1^{a_1} \cdots S_\sigma^{a_\sigma}}{D_1^{b_1} \cdots D_1^{b_\tau}} \right] = 0$$

"boundary" term @ infinity!

Example

1 loop tadpole

 $= \int \frac{d^Dk}{\pi^{D/2}} \frac{1}{(k^2+m^2)^n} = I(n)$

$\nearrow$ euclidean kinematics
no difference!

there is only 1 IBP

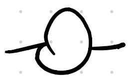
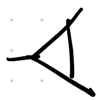
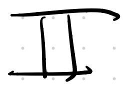
$$\int \frac{d^Dk}{\pi^{D/2}} \underbrace{\frac{\partial}{\partial k^\mu}\left(k^\mu \frac{1}{(k^2+m^2)^n}\right)} = 0$$

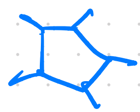
$$\int \frac{D}{k^2+m^2} - \int \frac{n \cdot k^\mu \cdot 2k_\mu}{(k^2+m^2)^{n+1}} = 0$$

$$\Rightarrow \int \frac{D-2n}{(k^2+m^2)^n} + 2n\, m^2 \int \frac{1}{(k^2+m^2)^{n+1}} = 0$$

$$\boxed{I(n+1) = -\frac{(D-2n)}{2n\,m^2}\, I(n)}$$

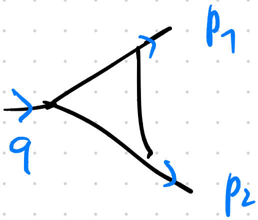
IBP allows to reduce any tadpole integral, to 1 master integral $I(1) =$ 

Similarly , ,  etc

BUT this procedure works in general "D", so it <u>DOES NOT</u> see that  → Boxes etc

in this sense, this procedure is "less physical", it does not capture simplifications which happen in $D=4$, also it mixes in general IR/UV properties in a non-trivial (and non-transparent) way.

For example



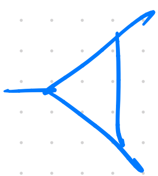
$$= \int \frac{d^D k}{\pi^{D/2}} \frac{1}{(k^2)^a \, ((k-p_1)^2)^b \, [(k-p_1-p_2)^2]^c}$$

massless props!

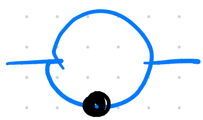
$= I(a,b,c)$ $\qquad p_1^2 = p_2^2 = 0 \; ; \; q^2 = s$

With IBPs one finds

$$I(1,1,1) = \frac{2}{D-4}\left[I(1,0,2) \right]$$



$= \frac{2}{D-4}$



"DOT"

IR singular

UV singular

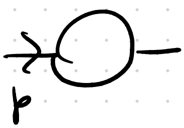
mixes two poles !

never get this with Integrand Reduction!

DIFFERENTIAL EQUATIONS

One of the most interesting consequences of IBPs is that we can use them to derive differential equations for master integrals

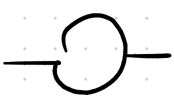
Let's consider 1 loop massive bubble



$$= \int \frac{d^D k}{\pi^{D/2}} \frac{1}{(k^2+m^2)^{n_1}\left((k-p)^2+m^2\right)^{n_2}} = f(p^2)$$

Euclidean kinematics, $p^2 = -s$; $s \to s + i\varepsilon$

IBP reduction shows that every integral can be written as linear combination of

 ;  = 2 MIs

Tadpole calculation is straight forward

$$I(1) = \int \frac{d^D k}{\pi^{D/2}} \frac{1}{k^2+m^2} = \frac{\Omega(D)}{\pi^{D/2}} (m^2)^{\frac{D-2}{2}} \int_0^\infty dk \, \frac{k^{D-1}}{1+k^2}$$

$$= 4 \, \Gamma\left(3 - \frac{D}{2}\right) \frac{(m^2)^{\frac{D-2}{2}}}{(D-2)(D-4)}$$

Let's consider bubble.

$$B(p^2, m^2) = \int \frac{d^D k}{\pi^{D/2}} \frac{1}{(k^2+m^2)\,((k-p)^2+m^2)}$$

$$= (m^2)^{\frac{D-4}{2}} \, B\left(\frac{p^2}{m^2}\right)$$ dimensional analysis

We can put $m^2 = 1$ for simplicity, reconstruct it at the end — $\Rightarrow B(p^2)$

now crucial point.

$$\frac{\partial}{\partial p^2} \Rightarrow \qquad p^2 = p^\mu p_\mu \to \frac{\partial p^2}{\partial p_\mu} = 2p^\mu \quad \text{so}$$

$$\frac{\partial}{\partial p_\mu} = \frac{\partial p^2}{\partial p^\mu}\frac{\partial}{\partial p^2} = 2p^\mu \frac{\partial}{\partial p^2}$$

$$= \quad p^\mu \frac{\partial}{\partial p^\mu} = 2p^2 \frac{\partial}{\partial p^2}$$

$$\frac{\partial}{\partial p^2} = \frac{1}{2p^2}\left[p^\mu \frac{\partial}{\partial p^\mu}\right]$$

We can write derivatives wrt mandelstam invariants as derivatives wrt momenta! p^μ

this means, I have an easy way to compute derivative of bubble at integrand level

$$\frac{\partial}{\partial p^2} B(p^2) = \frac{1}{2p^2} p_\mu \frac{\partial}{\partial p_\mu} \int \frac{d^D k}{\pi^{D/2}} \frac{1}{\underbrace{(k^2+m^2)}_{D_1} \underbrace{((k-p)^2+m^2)}_{D_2}}$$

$$\frac{\partial}{\partial p_\mu} \frac{1}{D_1} = 0$$

$$\frac{\partial}{\partial p_\mu} \frac{1}{D_2} = \frac{1}{D_2^2} 2(k-p)^\mu$$

so

$$p_\mu \frac{\partial}{\partial p_\mu} \frac{1}{D_2} = \frac{1}{D_2^2} \left(2k\cdot p - 2p^2 \right) =$$

$$= \frac{1}{D_2^2} \left(D_1 - D_2 - p^2 \right)$$

so finally we get

$$\frac{\partial}{\partial p^2}\int \frac{d^D k}{\pi^{D/2}}\frac{1}{D_1 D_2} = \frac{1}{2p^2}\int\frac{d^D k}{\pi^{D/2}}\left\{\frac{1}{D_2^2}-\frac{1}{D_1 D_2}-p^2\frac{1}{D_1 D_2^2}\right\}$$

$\Rightarrow$

$$\frac{\partial}{\partial p^2}\underbrace{I(1,1)}_{B(p^2)} = \frac{1}{2p^2}\Big[I(0,2)-I(1,1)\Big]$$

$$-\frac{1}{2}\, I(1,2)$$

IBPs

$$I(0,2) = -\frac{(D-2)}{2m^2}\overbrace{I(1,0)}^{T(m^2)}$$

$$I(1,2) = -\frac{(D-2)}{2m^2(p^2+4m^2)}\overbrace{I(1,0)}^{T(m^2)} - \frac{(D-3)}{p^2+4m^2}\underbrace{I(1,1)}_{B(p^2)}$$

putting everything together we get $(m^2=1)$

$$\frac{d}{dp^2} B(p^2) = \frac{1}{2}\left(\frac{D-3}{p^2+4} - \frac{1}{p^2}\right) B(p^2)$$

$$- \frac{D-2}{p^2(p^2+4)} \, T(1)$$

TADPOLE for $m^2=1$

→ inhomogeneous <u>linear</u> differential equation with **RATIONAL COEFFICIENTS**

⇑

<u>Important</u>: this is always the case, because IBPs generate always rational coefficients, nothing more complicated

let's solve it

1st] homogeneous part

$$\frac{d}{dp^2} f = \frac{1}{2}\left(\frac{D-3}{p^2+4} - \frac{1}{p^2}\right) f \Rightarrow f = \sqrt{\frac{(p^2+4)^{D-3}}{p^2}}$$

now redefine $B = f \cdot \bar{B}$

$$\frac{d\bar{B}}{dp^2} = -\sqrt{\frac{p^2}{(p^2+4)^{D-3}}}\,\frac{(D-2)}{p^2(p^2+4)}\,T(1)$$

$$\Rightarrow B(p^2) = \sqrt{\frac{(p^2+4)^{D-3}}{p^2}}\left[\int_0^{p^2} dt\left(-\frac{(D-2)}{t(t+4)}T(1)\sqrt{\frac{t}{(t+4)^{D-3}}}\right) + C\right]$$

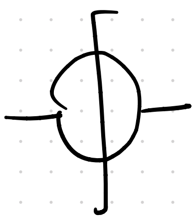
boundary condition

$$B(p^2) = -(D-2)\,T(1)\sqrt{\frac{(p^2+4)^{D-3}}{p^2}}\int_0^{p^2} dt\,\frac{t^{-1/2}}{(t+4)^{\frac{D-1}{2}}}$$

$$\underbrace{-(D-2)\,T(1)\sqrt{\frac{(p^2+4)^{D-3}}{p^2}}\cdot C}_{\text{Inferred from Boundary constant}}$$

here we can see immediately that $\underline{\underline{C=0}}$!

In fact, $B(p^2 \to 0)$ must be regular



$\Rightarrow$ no singular behavior except at $p^2 = -4m^2 \Rightarrow \underline{\underline{s = 4m^2}}$

$$B(p^2)\Big|_{p^2\to 0} \sim \frac{C}{\sqrt{p^2}} \to \infty \quad \text{so} \quad \underline{\underline{C=0}}\,!$$