12 - The Symbol Map for MPLs & special functions beyond MPLs

SS 2022

Lorenzo Tancredi

TUM

Let's go back to the problem of deriving FUNCTIONAL RELATIONS $\Rightarrow$ we'll do this in 2 ways, derivative & symbol

If we are interested into relations among functions of higher weight, we need to iterate this procedure

$$A = G(0,1,1,x)$$

$$B = -Li_2(x)\log(1-x) - \frac{1}{2}\log x \log^2(1-x)$$

$$+\frac{\pi^2}{6}\log(1-x) - Li_3(1-x) + \zeta_3$$

to prove $A = B$

$$\frac{\partial A}{\partial x} = \frac{1}{x} G(1,1,x) = \frac{1}{x}\frac{1}{2}\log^2(1-x)$$

$$\frac{\partial B}{\partial x} = +\frac{\log^2(1-x)}{2x} + \frac{Li_2(x)}{1-x} - \cancel{\frac{1}{2x}\log^2(1-x)} + \frac{\log(x)\log(1-x)}{1-x}$$

$$-\frac{\pi^2}{6}\frac{1}{1-x} + \frac{Li_2(1-x)}{1-x}$$

$$\frac{\partial A}{\partial x} = \frac{\log^2(1-x)}{2x}$$

$$\frac{\partial B}{\partial x} = \frac{\log^2(1-x)}{2x} + \frac{1}{1-x}\Big[\underbrace{Li_2(x) + Li_2(1-x) + \log x \log(1-x) - \frac{\pi^2}{6}}\Big]$$

For A, B to be equal, this should be zero !

$$\boxed{C = Li_2(x) + Li_2(1-x) + \log x \log(1-x) - \frac{\pi^2}{6}}$$

down to weight 2

so I need to consider now

$$\frac{\partial C}{\partial x} = 0$$

[obvious since we go down to logs !]

[used $\frac{\partial Li_2(x)}{\partial x} = -\frac{\log(1-x)}{x}$; $\frac{\partial Li_2(1-x)}{\partial x} = \frac{\log(x)}{1-x}$]

and notice $C(x \to \frac{1}{2}) = 0$

using $Li_2(\frac{1}{2}) = \frac{\pi^2}{12} - \frac{\log^2(2)}{2}$ etc this "multiple differentiation" can be encoded into the <u>SYMBOL MAP</u>

Start from formula for TOTAL DIFFERENTIAL of an <u>MPL</u> (proven on exercise sheet 6 !)

$$dG(a_1, \dots, a_n; x) =$$

$$\sum_{i=1}^{n} G(a_1, \dots, \hat{a}_i, \dots, a_n; x) \, d\log\left[\frac{a_{i-1} - a_i}{a_{i+1} - a_i}\right]$$

with $\begin{cases} a_i \neq a_j \text{ for all } i, j\,! \\ a_0 = x \,;\quad a_{n+1} = 0 \,;\quad \hat{a}_i \text{ missing index} \\ G(x) = 1 \text{ (without indices)} \,;\quad \boxed{dG(0, x) = d\log(x)} \end{cases}$

We define SYMBOL map $\mathcal{S}$ iteratively as

$$\mathcal{S}\left[G(a_1, \dots, a_n; x)\right] = \sum_{i=1}^{n} \mathcal{S}\left[G(a_1, \dots, \hat{a}_i, \dots, a_n; x)\right] \otimes \left[\frac{a_{i-1} - a_i}{a_{i+1} - a_i}\right]$$

$$\mathcal{S}\left[G(a,x)\right] = \mathcal{S}\left[G(x)\right] \otimes \operatorname{dlog}\left[\frac{x-a}{0-a}\right]$$

$$= \otimes(a-x) - \otimes a$$

$$= \otimes\left(1-\frac{x}{a}\right)$$

not well defined when $a \to 0$

$$\mathcal{S}\left[G(0,x)\right] = \mathcal{S}\left[G(x)\right] \otimes \operatorname{dlog} x = \underline{\underline{\otimes\, x}}$$

the symbol associates to any MPLs of weight n

a n-fold tensor in a tensor space

$$\mathcal{S}(G(0,x)) = \otimes x$$

$$\mathcal{S}(G(1,x)) = \otimes (1-x)$$

$$\underbrace{\mathcal{S}(G(-1,1,x)) = (1+x)\otimes 2 + (1-x)\otimes(1+x) - (1-x)\otimes 2}$$

⇑ ~ TOTAL DIFFERENTIAL MAXIMALLY ITERATED

Each factor in the symbol is "morally" a dlog and therefore the symbol can be simplified using standard properties of logarithms

$$a_1 \otimes \dots \otimes a_n \Rightarrow \text{each } a_i = \frac{da_i}{a_i} = d\log a_i$$

For classical polylogs

$$\mathcal{S}[\mathrm{Li}_n(x)] = -\left[(1-x)\otimes \underbrace{x\otimes\cdots\otimes x}_{n-1}\right]$$

note opposite order to integration

$$\boxed{\mathrm{Li}_n(x) = \int_0^x \frac{dx}{x}\,\mathrm{Li}_{n-1}(x) \qquad \mathrm{Li}_1(x) = -\log(1-x)}$$

$$d[\mathrm{Li}_n(x)] = \mathrm{Li}_{n-1}(x)\, d\log(x)$$

$$\mathcal{S}[\mathrm{Li}_n(x)] = \mathcal{S}[\mathrm{Li}_{n-1}(x)]\otimes x$$

$$= -(1-x)\otimes x\otimes\cdots\otimes x$$

PROPERTIES OF SYMBOL CALCULUS

$\Rightarrow$ unusual form of multilinearity

- **DISTRIBUTIVITY**

$$C \otimes (a \cdot b) \otimes D = C \otimes a \otimes D + C \otimes b \otimes D$$
$$C \otimes (a^n) \otimes D = n\, C \otimes a \otimes D$$

- **NEGLECT TORSION** (Technical "choice")

$$C \otimes \rho_n \otimes D = 0 \quad \text{if } \rho_n \text{ is n-th root of unity}$$

$\Rightarrow \log(1) = 0$!

- **ZERO ARGUMENT** $G(\ldots, 0) = 0 \Rightarrow \mathcal{S}[G(\ldots, 0)] = 0$

at least 1 index $\neq 0$

- **SHUFFLE PRODUCT**

$$\mathcal{S}[G(a_1 \ldots a_n; x)\, G(b_1 \ldots b_m, y)] =$$
$$\mathcal{S}[G(a_1 \ldots a_n, x)] ⧢ \mathcal{S}[G(b_1 \ldots b_m, y)]$$

shuffle tensors, see example later

. Note that we would expect $d\log c = 0$

for any constant c — instead one can extend symbol calculus to rational numbers following the definition we gave:

$$S[\log(2)] = \otimes\, d\log(2) = \underline{\underline{\otimes\, 2}} \neq 0!$$

$$S[Li_3(\tfrac{1}{2})] = 2 \otimes 2 \otimes 2 \quad \text{etc for } Li_n(\tfrac{1}{2})$$

(while $S[Li_n(2)] = 0$!)

since $S[Li_n(x)] = -(1-x) \otimes x \otimes \ldots \otimes x$

↑ if $x=2$, root of 1 !

$$S[Li_3(5)] = -2\,(2 \otimes 5 \otimes 5) \quad \text{etc}$$

now trivial wrt simply differentiating!

IMPORTANT PROBLEM

Many transcendental numbers are in the symbol kernel!

- ζ values $\underline{S[\zeta_n] = 0}$ $\Rightarrow$ because they are $\sim \underline{Li_n(1)}$

$$S(Li_n(x)) = -(1-x) \otimes x \otimes \cdots \otimes x$$

$x = 1$ gives zero due to $\log(1) = 0$

- Also some combinations

$$S\left[Li_4\left(\tfrac{1}{2}\right) + \tfrac{1}{24}\log^4 2\right] = 0 \quad \text{(check it!)}$$

MULTIPLE ζ values (MZVs)

$$\zeta_{m_1 \dots m_k} = \sum_{0<n_1<n_2<\dots<n_k} \frac{1}{n_1^{m_1} \dots n_k^{m_k}} = Li_{m_1 \dots m_k}(1)$$

Alternative Definition of MPLs

$$Li_{\underbrace{m_1 \dots m_k}}(x_1 \dots x_k) = \sum_{0<n_1<n_2<\dots<n_k} \frac{x_1^{n_1} \dots x_k^{n_k}}{n_1^{m_1} \dots n_k^{m_k}}$$

$$= (-1)^k \, G_{\underbrace{m_k \dots m_1}}\left(\frac{1}{x_k}, \dots, \frac{1}{x_1 \dots x_k}\right)$$

note reversed order!

where

$$G_{m_1 \dots m_k}(t_1, \dots, t_k) = G(\underbrace{0, \dots, 0}_{m_1 - 1}, t_1, \dots, \underbrace{0, \dots, 0}_{m_k - 1}, t_k; 1)$$

this implies $\mathcal{S}[\zeta_{m_1 \dots m_k}] = 0$ in kernel!

Let's go back to the problem we saw earlier

$a_0 = x$

$a_4 = 0$

$$A = G(0,1,1,x)$$

$$B = -Li_2(x)\log(1-x) - \frac{1}{2}\log x \log^2(1-x) + \frac{\pi^2}{6}\log(1-x) - Li_3(1-x) + \zeta_3$$

$$\begin{aligned} S(A) = {} & S(G(1,1,x)) \otimes \left(\frac{x-0}{1-0}\right) \\ & + S(G(0,1,x)) \otimes \left(\frac{0-1}{1-1}\right) = \\ & + S(G(0,1,x)) \otimes \left(\frac{1-1}{x-1}\right) = \end{aligned}$$

both divergent

need a regularization!

Divergence happens for equal successive indices.

Use shuffle!

$$G(1,1,x) = \frac{1}{2} G(1,x)^2$$

$$\mathcal{S}(G(1,x)) = \otimes(1-x)$$

$$\mathcal{S}(G(1,x)\,G(1,x)) = \mathcal{S}[G(1,x)] \sqcup\!\sqcup\, \mathcal{S}[G(1,x)] = 2(1-x)\otimes(1-x)$$

Let's do a "naive" regularization:

$$\mathcal{S}(G(1,1;x)) \Rightarrow \mathcal{S}[G(1,1+\varepsilon,x)] =$$

$$= (1-x)\otimes\varepsilon - \cancel{(1-x)\otimes(1+\varepsilon)} - \cancel{(1+\varepsilon)\otimes(1-x)}$$

$$+ \cancel{(1+\varepsilon)\otimes\varepsilon} + (1-x+\varepsilon)\otimes(1-x) - (1-x+\varepsilon)\otimes\varepsilon$$

if $\varepsilon \to 0$ $\quad (1+\varepsilon) = (1)$ root of unity! $[\log(1)=0]$

$$(1-x+\varepsilon) = (1-x)$$

$$= \underline{(1-x)\otimes\varepsilon} + (1-x)\otimes(1-x) - \underline{(1-x)\otimes\varepsilon} \quad \text{Cancel}$$

$$= (1-x)\otimes(1-x) \quad \text{regularized symbol!}$$

Similarly

$$\mathcal{S}\left[G(0,1,1;x)\right]_{reg} = (1-x)\otimes(1-x)\otimes x \quad !$$

then

$$B = -Li_2(x)\log(1-x) - \frac{1}{2}\log x\log^2(1-x) + \frac{\pi^2}{6}\log(1-x) - Li_3(1-x) + \zeta_3$$

$$\mathcal{S}[Li_2(x)] = -(1-x)\otimes x$$

$$\mathcal{S}[\log(1-x)] = \otimes(1-x)$$

$$\mathcal{S}[\log(x)] = \otimes x$$

$$\mathcal{S}[Li_3(1-x)] = -x\otimes(1-x)\otimes(1-x)$$

$$\mathcal{S}[\log x \log^2(1-x)] = \mathcal{S}[\log x \log^2(1-x)]$$

$$= 2\left[(1-x)\otimes(1-x)\otimes x + (1-x)\otimes x\otimes(1-x) + x\otimes(1-x)\otimes(1-x)\right]$$

$\uparrow$
2 is not inside the symbol !

Putting everything together one finds

$$\mathcal{S}\Big[-\mathrm{Li}_2(x)\log(1-x) - \frac{1}{2}\log(x)\log^2(1-x)$$

$$-\mathrm{Li}_3(1-x)\Big] = (1-x)\otimes(1-x)\otimes x$$

$$= \mathcal{S}\left[G(0,1,1,x)\right] \quad !$$

$$\left.\begin{aligned} &\mathcal{S}\left[\frac{\pi^2}{6}\log(1-x)\right] = 0 \\ &\mathcal{S}\left[\zeta_3\right] = 0 \end{aligned}\right\} \quad \text{For consistency :} \quad \underline{\mathcal{S}[\zeta_n] = 0}$$

we get a good part of the result, but not everything!

To fix last pieces, we need to refine our analysis …

One possible solution

Classify all terms in the kernel of S'

and make an ansatz $\Rightarrow$ coefficients can

be fixed by looking at special limits

Problem to find stuff like $\pi^2 \cdot \log(1-x)$

you need to "guess" also all possible functional forms.

it could be very difficult in general!

note that I would not lose this term using normal differential $\frac{\partial}{\partial x}$ so for some things S derivative is better!

$\Rightarrow$ this works well with HPLs, self-contained and very simple alphabet

A more complete solution

Use Coaction / Coproduct, which corresponds to not taking MAXIMUM ITERATION OF THE DIFFERENTIAL, but instead differentiating only "up to a point" —

the Coproduct reveals also the so-called HOPF-ALGEBRA Structure behind Polylogs

$\Rightarrow$ this would require another course —

if you are interested, check for example:

C. Duhr arXiv 1203.0454

S. Weinzierl arXiv 2201.03593

BEYOND MPLS

In constructing MPLs, we have started noticing that

1- Feynman Integrals fulfil differential Equations and are naturally solved in terms of iterated integrals

2- Feynman Integrals are <u>PERIODS</u>

3- Simplest example of iterated integrals that give rise to periods are iterated integrals of RATIONAL FUNCTIONS defined on the Riemann Sphere

$$\Downarrow$$

MPLS useful in <u>MANY problems!</u>

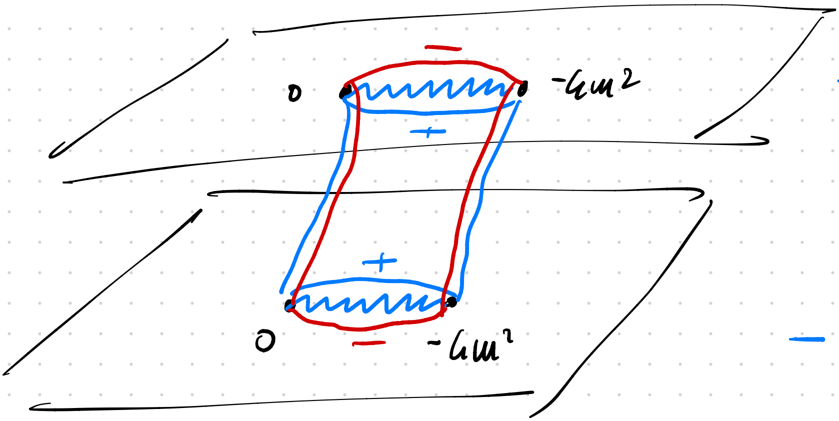
4- We know rational functions are not enough –

$\Longrightarrow$ MAX CUT of Feynman Ints might be more complicated

$D=2$

$$= \frac{1}{\sqrt{p^2(p^2+4m^2)}}$$

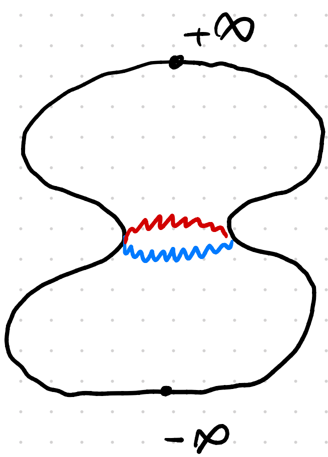
algebraic function



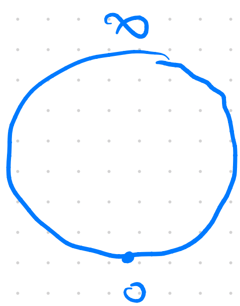
$+\sqrt{}$

$-\sqrt{}$

$\Downarrow$



$\cong$



topologically
$\sim$ Riemann sphere

indeed, there exist a change of variables that "straightens out" the branch cut

$$p^2 = m^2 \frac{(1-x)^2}{x} \qquad \sqrt{p^2(p^2+4m^2)} = m^2\left(\frac{x^2-1}{x}\right)$$

RATIONAL FUNCTION

→ if only 1 square root of degree up to 2 appears, we expect results to be still MPLs.

Note that $\sqrt{x-a}$, degree one

is the same as degree 2, sending one point to ∞

$$\underbrace{\int \frac{dx}{\sqrt{(x-a)(x-b)}}}_{\text{degree } 2} \quad \left(x \to \frac{1}{x}+b\right) = \int \frac{dx}{x^2} \frac{1}{\sqrt{\left(\frac{1}{x}+b-a\right)\left(\frac{1}{x}\right)}}$$

$$= \int \frac{dx}{x^{\cancel{2}}} \frac{\cancel{x}}{\sqrt{x(b-a)-1}}$$ one single root degree 1

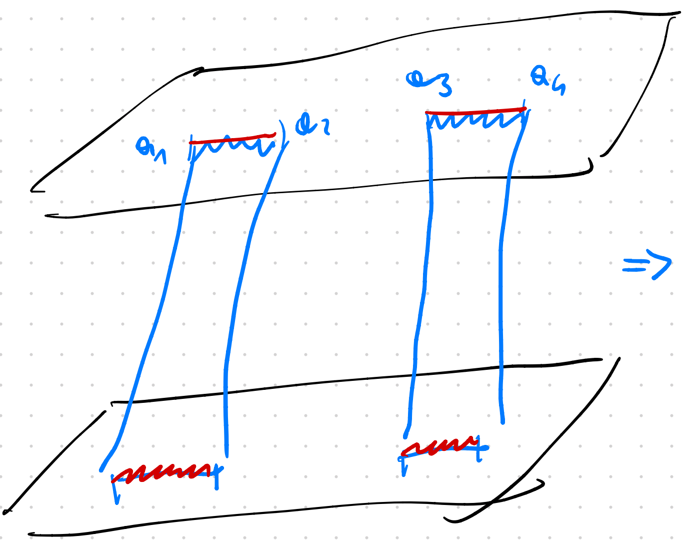
things start becoming interesting if $\sqrt{P_n(x)}$

degree 4 or 3

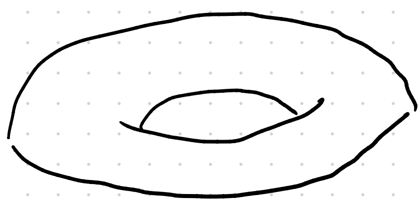
$\Rightarrow$ again the same thing, sending 1 point to infinity

General case degree 4

$$\sqrt{(x-a_1)(x-a_2)(x-a_3)(x-a_4)} = \sqrt{P_4(x)}$$



$\Rightarrow$



Torus $\sim$ surface of genus 1

the algebraic curve

$y^2 = P_4(x)$ is an Elliptic Curve

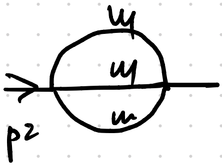
Elliptic Curve $\underset{\substack{\Uparrow \\ \text{Riemann} \\ \text{Surface}}}{\sim}$ Torus $(\mathbb{T})$

Repeat construction of (e)MPLs

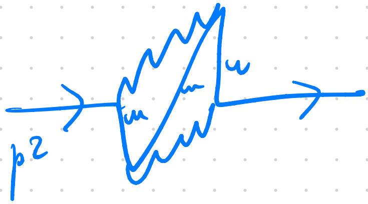
$\Rightarrow$ iterated integrals of "rational functions" on $\mathbb{T}$

$\mathcal{E}_4\begin{pmatrix} n_1 & \dots & n_k \\ a_1 & & a_k \end{pmatrix}; x\Big)$ Elliptic Polylogarithms

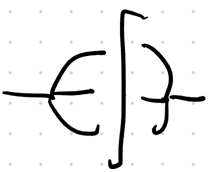
Are they relevant or just a mathematical curiosity?



two-loop sunrise graph, part of electron 2-loop propagator



it turns out that



$$= \int_{4m^2}^{(\sqrt{-p^2}-m)^2} db \, \frac{1}{\sqrt{b(b-4m^2)\left(b-(\sqrt{-p^2}-m)^2\right)\left(b-(\sqrt{p^2}+m)^2\right)}}$$

Elliptic Integral of the First Kind!

$\mathcal{E}_4$ needed for this 2-loop graph, & many more

Elliptic curve is first and simplest example of a Calabi-Yau surface $\Rightarrow$ relevant objects for String theory

TODAY'S HOT RESEARCH TOPIC:

a general theory of iterated integral on Calabi-Yau surfaces, using knowledge from STRING THEORY, to compute general Feynman Integrals "analytically"