11 - Multiple Polylogarithms

PART 2

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TUM

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MULTIPLE POLYLOGARITHMS & FUNCTIONAL RELATIONS

Recall definition of MPLs

$$G(a_1, \ldots, a_n; x) = \int_0^1 \frac{dt}{t-a_1} G(a_2, \ldots, a_n; t)$$

$a_j = \{0, 1, -1\}$

$\Uparrow$ SPECIAL CASE OF HPLs

$$G(\underbrace{0, 0, \ldots, 0}_{n}; x) = \frac{1}{n!} \log^n(x)$$

MPLs fulfil non-trivial functional relations if variable "x" appears with complicated functional dependence – moreover sometimes it is possible to express MPLs in terms of simpler functions

$\Rightarrow$ we'll see how to make sense of these things.

Fame of HPLs due to

- extreme symmetry, simplicity !
- wide applicability to diverse high-energy physics problems !

Remember canonical differential Equations satisfied by one loop bubble in $D \sim 2$

$$\frac{d\vec{m}}{dp^2} = \left(\frac{2-D}{2}\right) \begin{bmatrix} 0 & 0 \\ \frac{2}{\sqrt{p^2(p^2+4m^2)}} & \frac{1}{p^2+4m^2} \end{bmatrix} \vec{m}$$

Define now

$$p^2 = m^2 \frac{(1-x)^2}{x}$$

$$\frac{d}{dp^2} = \frac{dx}{dp^2} \frac{d}{dx} = \frac{x^2}{m^2(x^2-1)} \frac{d}{dx}$$

$\Rightarrow$

$$\frac{d\vec{m}}{dx} = \varepsilon \begin{bmatrix} 0 & 0 \\ \frac{2}{x} & \frac{2}{x+1} - \frac{1}{x} \end{bmatrix} \vec{m}$$

result is HPLs in x

x called <u>LANDAU VARIABLE</u>

if $0 < x < 1$ then $p^2 > 0$ (Euclidean Kinematics!)

at every order result written as

$$\int dx \left\{ \frac{1}{x} ; \frac{1}{x+1} \right\} G(\vec{a} ; x)$$

$\vec{a} = \{0, -1\}$ HPLs

Explicitly for bubble, m_2

$$\frac{dm_2^{(0)}}{dX} = 0 \quad \Rightarrow \quad m_2^{(0)} = C_0$$

boundary cond.

$$\frac{dm_2^{(1)}}{dX} = \frac{2}{X} m_1^{(0)} + \left(\frac{2}{X+1} - \frac{1}{X}\right) C_0$$

TADPOLE

in our normalization

$$m_1 = 1 + \varepsilon^2 \frac{\pi^2}{12} - \varepsilon^3 \frac{\zeta_3}{3} + O(\varepsilon^3) \qquad \varepsilon = \left(\frac{2-D}{2}\right)$$

so

$$m_2^{(1)} = 2\,G(0,x) + C_0\left[2\,G(-1,x) - G(0,x)\right] + C_1$$

must be fixed still!

We can use regularity, remember

$$m_2 = \left(\frac{2-D}{2}\right)\sqrt{p^2(p^2+4m^2)}\; M_2$$

$\Uparrow$ physical integral

must be regular at $p^2=0$

$$M_2 = \frac{1}{\varepsilon}\frac{1}{\sqrt{}}\, m_2 = \frac{1}{\varepsilon}\frac{x}{m^2(x-1)(x+1)}\, m_2$$

$p^2 \to 0$ corresponds to $x \to 1$ $\left(p^2 = m^2\frac{(1-x)^2}{x}\right)$

so pole $\frac{1}{x-1}$ must disappear!

$$M_2^{(0)} = \frac{1}{\varepsilon}\frac{x}{m^2(x-1)(x+1)}\, c_0 \overset{x\to 1}{\sim} \frac{c_0}{x-1} \quad \text{so } c_0 \equiv 0$$

$$M_2^{(1)} = \frac{x}{m^2(x-1)(x+1)}\left[2G(0,x) + c_1 + \cancel{\overset{\overset{0}{\shortparallel}}{c_0}\left(2G(-1,x) - G(0,x)\right)}\right]$$

∞ $x \to 1$ $\quad G(0,x) = \log(x) \sim (x-1)$!

$$M_2^{(1)} \sim \frac{1}{x-1}\left[\underset{\substack{\shortparallel}}{c_1} + \cancel{(x-1)} + \cdots\right]$$

must be zero too !

$\leadsto$ regularity tells us that $\underline{c_0 = c_1 = 0}$

$$m_2^{(0)} = 0$$

$$m_2^{(1)} = 2\,G(0,x) = \underline{2\log(x)}$$

pretty simple!

$$\frac{dm_2^{(2)}}{dx} = \frac{2}{x}\overset{\overset{0!}{\shortparallel}}{m_1^{(1)}} + \left(\frac{2}{x+1} - \frac{1}{x}\right) m_2^{(1)}$$

$$= \underline{\left(\frac{2}{x+1} - \frac{1}{x}\right) 2\,G(0,x)}$$

$$m_2^{(2)} = 4\,G(-1,0,x) - 2\,G(0,0,x) + C_2$$

still to be fixed.

again I want $M_2^{(2)} \sim \frac{1}{x-1}\, m_2^{(2)} \to$ finite as $x \to 1$

$G(0,x) \sim (x-1)$ $\quad G(0,0,x) \sim (x-1)^2$

$$G(-1,0,x) = \int_0^x \frac{dt}{t+1} \log(t) \quad \text{when } x \to 1$$

$$G(-1,0,1) = -\frac{\pi^2}{12} = -\frac{\zeta_2}{2}$$

so $M_2^{(2)} \underset{x\to 1}{\sim} \frac{1}{x-1}\left[C_2 + \zeta_2\right] \Rightarrow C_2 = -\zeta_2$

Finally we get

$$\underbrace{m_2^{(2)} = 4G(-1,0,x) - 2G(0,0,x) - \zeta_2}_{\text{only functions and numbers of transcendental weight 2}}$$

true at every order in $\varepsilon \Rightarrow \varepsilon^n = \text{weight } n$

We say that integrals that fulfil canonical differential equations are also of

UNIFORM TRANSCENDENTAL WEIGHT (UT)

$0 < x < 1 \Rightarrow$ Euclidean $p^2 > 0 \Rightarrow s < 0$

FUNCTIONAL RELATIONS & LINEAR INDEPENDENCE

If we think about MPLs as special case of iterated integrals, we see that statement on linear independence applies here straightforwardly —

For MPLs $\omega_i = d\log(x - a_i)$

if all a_i different, then clearly all ω_i are independent

$\Rightarrow \nexists\, f$ RATIONAL FUNCTION such that

$$\sum_i b_i \omega_i = df$$

↳ by constructions, MPLs evaluated at generic point "x" are linearly independent if they 'look' different $\Rightarrow$ after applying all SHUFFLES.

For linear independence it is FUNDAMENTAL that MPLs are function of the same variable (x) and that x does not enter in the indices!

$$G(3,2,0,x) \neq G(2,1,0,x) \quad \text{etc}$$

if instead G evaluated at some special numerical point, this is in general not true!

Also $G(0,1;x)$ cannot be easily compared with $G(0,1,1-x)$

or any non-trivial function of x

$\Downarrow$

they might in general not be independent $\Rightarrow$ FUNCTIONAL RELATIONS

We know this very well with normal logs

$$\log(x) = -\log\left(\frac{1}{x}\right)$$

$$\Rightarrow G(0,x) = -G\left(0,\frac{1}{x}\right)$$

SOMETIMES functional relations involve crossing a branch cut!

$$G\left(1,\frac{1}{x}\right) = \log\left(1-\frac{1}{x}\right)$$

if $x>1$ ok

if $0<x<1$ branch cut!

Keep track of it with $i\varepsilon$ $x \to x+i\varepsilon$

$$\log\left((x+i\varepsilon-1)/(x+i\varepsilon)\right) = \log(x-1+i\varepsilon) - \log(x+\cancel{i}\varepsilon)$$

$$= \log(1-x) + \log(-1+i\varepsilon) - \log(x)$$

$$= G(1,x) - G(0,x) \underline{\underline{+i\pi}}$$ imaginary part

<u>Important point</u>: when dealing with MPLs

if we understand functional relations for WEIGHT 1 functions (LOGS!)

we can derive ALL FUNCTIONAL RELATIONS of any weight, <u>iteratively</u>!

for example we have seen

$$\begin{cases} G\left(1, \frac{1}{x}\right) = G(1,x) - G(0,x) + i\pi \\ G\left(0, \frac{1}{x}\right) = -G(0,x) \end{cases} \quad \text{if} \quad \boxed{\begin{matrix} 0 \le x \le 1 \\ \mathrm{Im}(x) > 0 \end{matrix}}$$

From here, we can derive similar relations for

$$G\left(0, 1, \frac{1}{x}\right) = \int_0^{1/x} \frac{dt}{t} G(1,t)$$

$$\frac{\partial}{\partial x} G\left(0,1,\frac{1}{x}\right) = -\frac{1}{x^2}\frac{1}{\left(\frac{1}{x}\right)} G\left(1,\frac{1}{x}\right)$$

$$= -\frac{1}{x} G\left(1,\frac{1}{x}\right)$$

$$= \frac{1}{x}G(0,x) - \frac{1}{x}G(1,x) - \frac{1}{x}i\pi \quad \boxed{\text{if } \begin{array}{l} 0\le x<1 \\ \operatorname{Im}(x)>0 \end{array}}$$

so integrating back

$$G\left(0,1,\frac{1}{x}\right) = \int^{x} dx \left[\frac{1}{x}G(0,x) - \frac{1}{x}G(1,x) - \frac{i\pi}{x}\right] + C$$

$$= G(0,0,x) - G(0,1,x) - i\pi\, G(0,x) + C$$

constant to be fixed!

lim $x \to 1$

$$G(0,1,1) = -G(0,1,1) + C \Rightarrow C = 2\,G(0,1,1)$$

$$G(0,1,1) = \int_0^1 \frac{dx}{x} \log(1-x) = -\mathrm{Li}_2(1) = -\frac{\pi^2}{6}$$

$$\underbrace{G\left(0,1,\frac{1}{x}\right)}_{\text{weight } 2} = \underbrace{G(0,0,x) - G(0,1,x) - \frac{\pi^2}{3} - i\pi\, G(0,x)}_{\text{all terms of weight 2 !}}$$

CONJECTURE :

It is conjectured that ALL FUNCTIONAL RELATIONS among MPLs preserve transcendental weight

IMPORTANT : by differentiating an MPL wrt any variable (INCLUDING INDICES!) we obtain MPLs of LOWER WEIGHT.

in this way we can work out ANY functional relation bottom-up, starting from LOGARITHMS

in other words, if we understand functional relations for logarithms, we understand them for ANY MPLs

Let us consider the two following quantities $(0<x<1)$

$$A = G(-1,1,x)$$

$$B = \left[-\mathrm{Li}_2\left(\frac{1+x}{2}\right) + \log 2 \log(1+x) - \frac{\log^2 2}{2} + \frac{\pi^2}{12}\right]$$

it turns out $A = B$

we can prove it by differentiation

$$\frac{\partial A}{\partial x} = \frac{1}{x+1} G(1,x) = \frac{\log(1-x)}{1+x}$$

they match!

$$\frac{\partial B}{\partial x} = + \frac{\log((1-x)/2)}{1+x} + \log(2)\frac{1}{1+x} = \frac{\log(1-x)}{1+x}$$

then when $x \to 0$ $\begin{cases} A(0) = 0 \\ B(0) = -\frac{\pi^2}{12} + \frac{\log^2 2}{2} - \frac{\log^2 2}{2} + \frac{\pi^2}{12} = 0 \end{cases}$ ✓

If we are interested into relations among functions of higher weight, we need to iterate this procedure

$$A = G(0,1,1,x)$$

$$B = -\mathrm{Li}_2(x)\log(1-x) - \frac{1}{2}\log x \log^2(1-x)$$

$$+ \frac{\pi^2}{6}\log(1-x) - \mathrm{Li}_3(1-x) + \zeta_3$$

to prove $A = B$

$$\frac{\partial A}{\partial x} = \frac{1}{x}G(1,1,x) = \frac{1}{x}\frac{1}{2}\log^2(1-x)$$

$$\frac{\partial B}{\partial x} = + \frac{\log^2(1-x)}{2x} + \frac{\mathrm{Li}_2(x)}{1-x} - \cancel{\frac{1}{2x}\log^2(1-x)} + \frac{\log(x)\log(1-x)}{1-x}$$

$$- \frac{\pi^2}{6}\frac{1}{1-x} + \frac{\mathrm{Li}_2(1-x)}{1-x}$$

$$\frac{\partial A}{\partial x} = \frac{\log^2(1-x)}{2x}$$

$$\frac{\partial B}{\partial x} = \frac{\log^2(1-x)}{2x} + \frac{1}{1-x}\underbrace{\left[Li_2(x) + Li_2(1-x) + \log x \log(1-x) - \frac{\pi^2}{6} \right]}$$

1- to be equal, this should be zero!

$$C = Li_2(x) + Li_2(1-x) + \log x \log(1-x) - \frac{\pi^2}{6}$$

so I need to consider now

$$\frac{\partial C}{\partial x} = 0$$ [obvious since we go down to logs!]

[used $\frac{\partial Li_2(x)}{\partial x} = -\frac{\log(1-x)}{x}$; $\frac{\partial Li_2(1-x)}{\partial x} = \frac{\log(x)}{1-x}$]

and notice $C(x \to \frac{1}{2}) = 0$

using $Li_2(\frac{1}{2}) = \frac{\pi^2}{12} - \frac{\log^2(2)}{2}$ etc