10 - Multiple Polylogarithms

part 1

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In previous lecture we have seen how Feynman Integrals, through differential equations, can naturally be expressed in terms of ITERATED INTEGRALS.

$\Rightarrow$ At least their ε-expansion!

Iterated integrals are usefull, but they are a huge class of functions, and it turns out that in practical applications, some special SUBSETS of them are most relevant

Specifying a subset amounts to specifying the type of differential forms

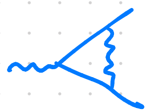
The most important (and historically the FIRST) type of iterated integrals that was studied for its applications to particle physics, are the so called

MULTIPLE POLYLOGARITHMS

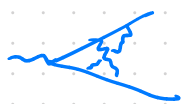
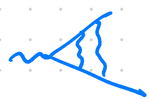
Historically, from the very first days of perturbative QFT, physicists started noticing that some special types of functions and numbers would appear over and over again in perturbative calculations —

typical examples:

g-2 electron

$$a = \frac{g-2}{2} = \frac{1}{2}\left(\frac{\alpha}{\pi}\right) \sim$$ 

$$+ \left[\frac{197}{144} + \frac{\pi^2}{12} - \frac{\pi^2 \ln 2}{2} + \frac{3}{4}\zeta_3\right]\left(\frac{\alpha}{\pi}\right)^2$$

$\hookrightarrow$   etc

$$+ C_3\left(\frac{\alpha}{\pi}\right)^3 + \dots$$

$$C_3 \supset \left\{\pi, \zeta_3, \zeta_5, \ln 2, \mathrm{Li}_4\left(\tfrac{1}{2}\right) \leftarrow \text{new}\right\}$$

Vacuum polarization in QED

 $= \left[g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right] \Pi(p^2, m^2)$

$$\Pi(p^2, m^2) > \frac{1}{\sqrt{p^2(p^2+4m^2)}} \ln\left(\frac{\sqrt{p^2+4m^2} - \sqrt{p^2}}{\sqrt{p^2+4m^2} + \sqrt{p^2}} \right)$$

we see logarithms, π, ζ_n zeta values, etc...

$$\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n} \; ; \quad k > 1$$

which for even "n" gives

$$\zeta_{2n} = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}$$

B_{2n} = Bernoulli Numbers

$B_2 = \frac{1}{6}$; $B_4 = -\frac{1}{30}$; $B_6 = \frac{1}{42}$

In $g-2$ we also encounter classical polylogarithms

$$\mathrm{Li}_n(x) = \int_0^x dt \, \frac{\mathrm{Li}_{n-1}(t)}{t} \qquad \mathrm{Li}_1(t) = \log(1-t)$$

$$\mathrm{Li}_n(1) = \zeta_n \qquad n > 1$$

Notice that arguments of log in $\mathcal{O}(m)$ is "complicated" algebraic function!

Can we make some order?

- which functions are "allowed to appear"?
- which argument? can I get $\log(\log x)$?
- what about e, γ_E, $\log(\pi)$ etc, which you see in QFT books? Are they really there?

First of all we need to define an important property of numbers (and functions, in some extended sense) => <u>TRANSCENDENTALITY</u>

- A number (in $\mathbb{C}$) is algebraic (over $\mathbb{Q}$) if it is the root of some polynomial with coefficients in $\mathbb{Q}$

  - $q \in \mathbb{Q}$ algebraic => $z - q = 0$
  - $q \in \mathbb{Q}$ => $\sqrt[n]{q}$ algebraic, $z^n - q = 0$
  - $i \in \mathbb{C}$ is algebraic $x^2 + 1 = 0$

- A number that is <u>NOT</u> algebraic, is called TRANSCENDENTAL.

Algebraic numbers form a <u>field</u>!

$a+b$; $a \cdot b$; $\frac{1}{a}$ still algebraic if a, b are algebraic a <u>FUNCTION</u> is algebraic if it is a root of a polynomial with coefficients that are RATIONAL FUNCTIONS

$$(x+1)\,z^2 - 3x = 0$$

$$z = \pm\sqrt{\frac{3x}{x+1}}$$ algebraic function!

it is in general VERY DIFFICULT to prove that a number or a function is <u>TRANSCENDENTAL</u>.

There is a theorem [LINDEMANN, WEISTRASS - GELFOND]

- if $y, z \in \mathbb{C}$, algebraic, $y \neq 0,1$; z not rational
  $\Rightarrow$ y^z is transcendental
- e is transcendental [$e^1 = e$; 1 algebraic]
- π is transcendental $e^{i\pi} = -1$; $\left\{\begin{matrix} -1 \\ i \end{matrix}\right\}$ both algebraic

- π^n (and so ζ_{2n}) are transcendental
- e^{π} Gelfond's constant $\Rightarrow (e^{i\pi})^{-i} = (-1)^{-i}$
- $\log x$ is transcendental since $e^{\log x} = x$ is algebraic!

Note that ζ_3 is the only odd ζ value that was proven to be transcendental, $\zeta_{5,7,9}$ etc just conjecture

CONJECTURE ζ_n and all $\mathrm{Li}_n(x)$ are transcendental!

it looks like both algebraic and transcendental objects appear in Feynman integrals. There exist another class of objects that we can define, and that has non-trivial intersection with the previous ones:

PERIODS

A complex number is a period if $z = x + iy$ s.t x, y can both be written as integrals of algebraic functions with algebraic coefficients over a domain defined by polynomial inequalities with algebraic coefficients

- Every algebraic number is a period

$$x = \int_0^x dy$$

- the logarithm is a period

$$\log(x) = \int_1^x \frac{dy}{y}$$

- π is a period

$$\pi = \int_{x^2+y^2 \leq 1} dx\,dy$$

- All (classical) polylogs are periods

$$\mathrm{Li}_2(z) = \int_{0 \leq t_2 \leq t_1 \leq z} \frac{dt_1 dt_2}{t_1(1-t_2)}$$

- $\boxed{\cdot}$ γ_E, $\sin x$, $\cos x$, $\log(\log x)$ are CONJECTURED not to be periods!
[$\arcsin x$ is! it's a log!]

- **IMPORTANT THEOREM** (Bogner, Weinzierl 2009)

if you properly normalise them, Feynman integrals are periods

$$I^{(L)} = e^{L \gamma_E \varepsilon} \int \prod_{j=1}^{L} \frac{d^D k_j}{i \pi^{D/2}} \frac{1}{D_1 \cdots D_N}$$

this normalization removes ALL occurrences of γ_E

the last theorem motivates us to consider iterated integrals that produce "periods" — simplest type of "algebraic" functions that we can imagine to integrate, are RATIONAL FUNCTIONS

$R(x, \vec{y})$ = rational function

$\uparrow$ set of arbitrarily many variables

By partial fractioning in x, clearly we have

$$\int R(x,\vec{y})\, dx = \begin{cases} \int dx\, x^n = \frac{1}{n+1} x^{n+1} \\ \int dx \frac{1}{[x-a(\vec{y})]^n} = -\frac{1}{n-1} \frac{1}{[x-a(y)]^{n-1}}\,; & \underline{\underline{n>1}} \\ \int dx \frac{1}{x-a(\vec{y})} = \log(x-a(\vec{y}))\,; & n=1 \end{cases}$$

we see that integrating a rational function we get something other than a rational function only if we integrate over a single pole

$\Longrightarrow$ residue $\neq 0$, multivariate function called logarithm.

it is very natural to extend this construction allowing iterated integrations

$$\int_0^z \frac{1}{(x-a)^n}\, dx \int_0^x dy \left(\frac{1}{y-b}\right) =$$

if $n>1$, can do integration by parts

$$= -\frac{1}{n-1}\frac{1}{(x-a)^{n-1}} \underbrace{\int_0^x dy \frac{1}{y-b}} + \frac{1}{n-1} \underbrace{\int_0^z \frac{1}{(x-a)^{n-1}}\left(\frac{1}{x-b}\right) dx}$$

again just log + rational functions!

again, only if $n = 1$, I get something "new"

$$\int_0^z \frac{dx}{x-a} \int_0^x \frac{dy}{y-b} \equiv G(a,b,z)$$

First example of a multiple polylog

DEFINITION MPLs

$$\boxed{G(a_1, a_2, \dots, a_n; x) = \int_0^x \frac{dt}{t-a_1} G(a_2, \dots, a_n; t)}$$

see that iterative definition has a problem when

$a_1 = \dots = a_n = 0 \Rightarrow$ tangential base point regularization

or simply

$$\boxed{G(\underbrace{0, \dots, 0}_{n}, x) = \frac{1}{n!} \log(x)^n}$$

- MPLs are iterated integrals, whose differential forms are
  - dlogs$(f(x))$
  - where $f(x)$ is a RATIONAL FUNCTION

- $G(\underbrace{0,\dots,0}_{n};x) = \frac{1}{n!}\log^n(x)$ consistent with

$$= \int_1^x \frac{dt_1}{t_1} \cdots \int_1^{t_{n-1}} \frac{dt_n}{t_n}$$

$\vec{a} = a_1, \dots, a_n$ called indices of polylog

n = length $\stackrel{!}{\equiv}$ transcendental weight

$\Uparrow$

SPECIAL FOR POLYLOGS!

$$G(a,x) = \int_0^x \frac{dt}{t-a} = \log\left(1-\frac{x}{a}\right) = \text{transcendental function of weight } 1$$

$$\cdot\ G(a,a,\ldots,a,x) = \frac{1}{n!}\log\left(1-\frac{x}{a}\right)^n$$

product gives transcendental weight "n"

· As every iterated integral, they fulfil all properties including SHUFFLE PRODUCT

$$G(a,x)\,G(b,x) = G(a,b,x) + G(b,a,x)$$

$$\begin{aligned} G(a,b,x)\,G(c,d;x) = {} & G(a,b,c,d;x) + G(a,c,b,d,x) \\ & + G(a,c,d,b,x) + G(c,a,b,d,x) \\ & + G(c,a,d,b,x) + G(c,d,a,b,x) \text{ etc} \end{aligned}$$

consistent with

$$\underbrace{G(a,x)\cdot\ldots\cdot G(a,x)}_{n \text{ times}} = n!\; G(\underbrace{a,a,\ldots,a}_{n\text{ times}};x)$$

$$\log\left(1-\frac{x}{a}\right)^n = n!\; G(a,\ldots,a;x)$$

OTHER BASIC PROPERTIES

1) $G(a_1, \ldots, a_n; x)$ is logarithmically divergent when $x \to a_1$

2) $G(a_1, \ldots, a_n; x)$ is smooth at $x = 0$ if $a_j \neq 0$ and $\underline{G(a_1, .., a_n; 0) = 0}$

3) $G(a_1, \ldots, a_n; x)$ as a function of $x \in \mathbb{C}$, has a branch cut AT MOST if $Re(x) > Re(a_i)\ \forall i$ but not necessarily!

$G(a, x) = \log\left(1 - \frac{x}{a}\right)$ has branch cut if $x > a$

$G(0, 1, x) = \int_0^x \frac{dt}{t} \int_0^t \frac{du}{u-1} = -Li_2(x)$ branch from $Re(x) > 1$

4) Rescaling : if $\boxed{a_n \neq 0}$, then

$$G(\lambda a_1, \dots, \lambda a_n ; x) = G\left(a_1, \dots, a_n ; \frac{x}{\lambda}\right)$$

5) CLASSICAL POLYLOGARITHMS are a special case

$$\mathrm{Li}_n(x) = -G(\underbrace{0, 0, \dots 0,}_{n-1} 1, x)$$

$$\mathrm{Li}_2(x) = -G(0, 1, x)$$

etc

$$\mathrm{Li}_3(x) = -G(0, 0, 1, x)$$

6) HARMONIC POLYLOGARITHMS (HPLs) most famous HPLs

fix indices $= \{0, 1, -1\}$

Remiddi, Vermaseren '99