Central Tutorial - 19/4/18

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1 Interacting non-abelian gauge theories (P&S 16.1)

The goal of this section is to derive the Feynman rules of non-abelian gauge theories (following P&S section 16.1). This would be useful for several reasons, mainly

1. Calculating observables: Feynman rules are a recipe for the calculation of Feynman diagrams. Feynman diagrams are needed for the calculation of scattering amplitudes, which can eventually be using to calculate cross sections.
2. Understanding the gauge structure of non-abelian theories : in gauge theories, some degrees of freedom are unphysical, in the sense that they are gauge-dependent, and we identify systems which are related by gauge symmetry as being physically equivalent (gauge symmetry=redundancy). E.g as was shown in class, in an abelian gauge theory, the starting point is a vector field with 4 degrees of freedom. One degree of freedom can be removed using the E.O.M, and an additional degree of freedom can be removed by fixing a gauge and thus picking a particular description out of an infinite set of equivalent descriptions related by gauge transformations. For an abelian theory, the cancelation of the non-physical degrees of freedom (=transverse photon polarization) is insured by the Ward identity. For a non-abelian theory, this cancelation is more complicated and requires the introduction of additional fields called ghosts.
3. Calculating the β-function : the coupling in gauge theories changes with the energy scale, and the energy dependence is given by the β-function. For example, QCD is a non-abelian gauge theory, and its coupling gets weaker in higher energies (=smaller distances), and stronger in lower energies (=larger distances). The latter leads to the phenomena of confinement.

Let us begin with deriving the Feynman rules for the Yang-Mills Lagrangian.

$$\mathcal{L} = -\frac{1}{4}F^{a\mu\nu}F_{a\mu\nu} + \bar{\psi}(i\not{D} - m)\psi\,. \tag{1}$$

with a a group index and ψ in some irreducible representation r of the gauge group $\mathcal{G}$ (group indices in last term are suppressed). We have the field strength

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu\,. \tag{2}$$

f^{abc} are the structure constants of $\mathcal{G}$. The covariant derivative

$$D_\mu = \partial_\mu - igA^a_\mu t^a_r\,. \tag{3}$$

We can split out Lagrangian to the free and interacting parts

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_{\text{int}}\,,\\
\mathcal{L}_0 &= \frac{1}{2}A_\mu\left(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)A_\nu + \bar{\psi}(i\not{\partial} - m)\psi\,,\\
\mathcal{L}_{\text{int}} &= -gf^{abc}(\partial_\lambda A_{a\sigma})A^\lambda_b A^\sigma_c - \frac{1}{4}g^2 f^{eab}f^{ecd}A^\lambda_a A^\sigma_b A_{c\lambda}A_{d\sigma} + gA^a_\lambda(\bar{\psi}t^a_r\gamma^\lambda\psi)\,.
\end{aligned} \tag{4}$$

Our free theory consists of $d(r)$ Dirac fermions and $d(G)$ gauge fields. We can solve the free theory exactly by performing the gaussian integration. The building block of the free theory are the propagators (or 2-point functions)

$$\langle \psi_{i\alpha}(x)\bar{\psi}_{j\beta}(y)\rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\left(\frac{i}{\not{k}-m}\right)_{\alpha\beta}\delta_{ij}e^{-ik(x-y)}\,, \tag{5}$$

$$\langle A_{i\mu}(x)A_{j\nu}(y)\rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\left(\frac{-i\eta_{\mu\nu}}{k^2}\right)\delta_{ij}e^{-ik(x-y)}\,, \tag{6}$$

with which we can calculating all n-points functions in the free theory using Wick's theorem. Now we can write down the Feynman rules for the interaction vertices. The Feynman rule for the fermion gauge interaction is straightforward. For the gauge-fields self-interaction some care is required. First we must label our external legs with Lorenz and group index and fix a momentum convention, in out case we take all momenta to be points inwards. For example, we calculate the 3 gauge boson vertex $V^{(3)}$ be taking the variational derivative of the action in momentum space

$$(2\pi)^4\delta^4(k+p+q)\times V^{(3)}{}^{\mu\nu\rho}_{abc}(k,p,q) \equiv i\left(\frac{\delta S}{\delta A^a_\mu(k)A^b_\nu(p)A^c_\rho(q)}\right)\,, \tag{7}$$

with

$$S = \int\prod_{i=1}^{3}\frac{\mathrm{d}^4 k_i}{(2\pi)^4}(2\pi)^4\delta^4\left(\sum_{i=1}^{3}k_i\right)\left[igf^{abc}k_{1\lambda}A_{a\sigma}(k_1)A^\lambda_b(k_2)A^\sigma_c(k_2)\right]\,,$$
$$\frac{\delta A_{a\mu}(k_1)}{\delta A^\nu_b(k_2)} = \delta_{ab}\eta_{\mu\nu}(2\pi)^4\delta^4(k_1-k_2)\,. \tag{8}$$

For the 3 gauge boson vertex, there are 3! possible contractions leading to 6 distinct terms in the vertex. For the 4 gauge boson vertex, there are 4! possible contractions. We can group these 24 contraction into 6 group of equal terms, leading to 6 distinct terms in the vertex and the cancelation of the $\frac{1}{4}$ factor. Similarly to the abelian theory,

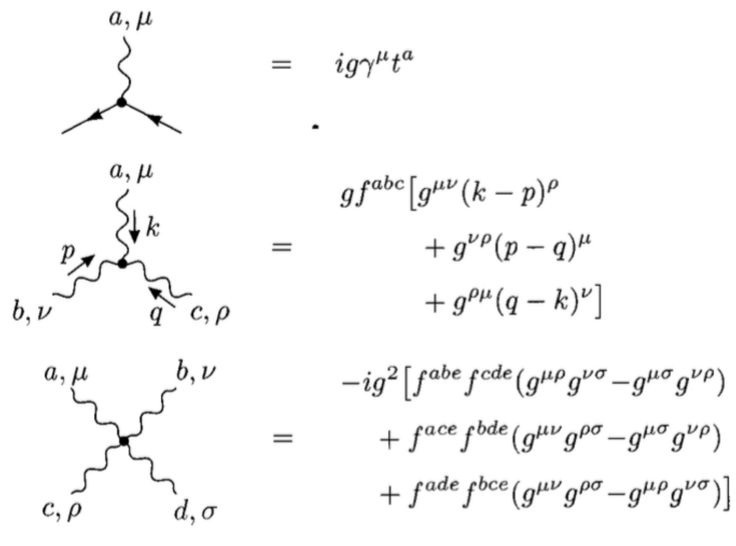

Figure 16.1. Feynman rules for fermion and gauge boson vertices of a non-Abelian gauge theory.

we know that the non-abelian gauge boson has only 2 physical polarization states, and we expect that physical processes cannot produce a non-physical polarization state. Therefore the physical amplitude $\mathcal{M}^\mu(k,...)$ (=with on shell external states) with an external gauge boson with momentum k^μ should satisfy

$$k_\mu\mathcal{M}^\mu(k,...) = 0\,. \tag{9}$$

This relation can also be understood in the following way: one can perform a little group transformation Λ^μ_ν, which by definition leaves the momentum invariant $p^\mu = \Lambda^\mu_\nu p^\nu$, but shifts the polarization vector by a vector proportional to the momentum $\epsilon^{'\mu} = \Lambda^\mu_\nu \epsilon^\nu = \epsilon^\mu + cp^\mu$. Lorentz invariance requires

$$\epsilon^\mu \mathcal{M}_\mu = \epsilon'^\mu \mathcal{M}'_\mu = (\epsilon^\mu + cp^\mu)\Lambda^\nu_\mu \mathcal{M}_\nu = (\epsilon^\nu + 2cp^\nu)\mathcal{M}_\nu \quad \to \quad p^\mu \mathcal{M}_\mu = 0\,. \tag{10}$$

Let us check the ward identity in a simple case in a non-abelian theory, looking at a fermion-anti-fermion pair scattering into two gauge bosons. The first two diagrams give us

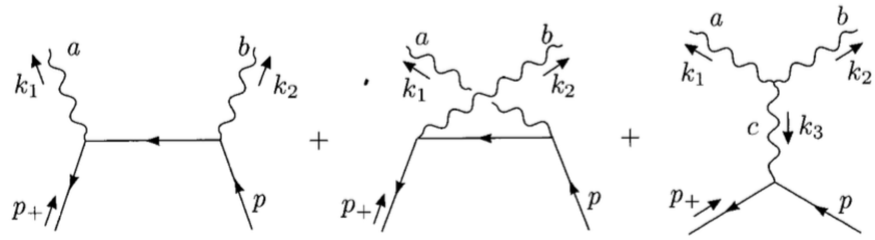

Figure 16.2. Diagrams contributing to fermion-antifermion annihilation to two gauge bosons.

$$i\mathcal{M}^{\mu\nu}_{1,2}\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2) = (ig)^2\bar{v}(p_+)\left[\gamma^\mu t^a \frac{i}{\not{p} - \not{k}_2 - m}\gamma^\nu t^b + \gamma^\nu t^b \frac{i}{\not{k}_2 - \not{p}_+ - m}\gamma^\mu t^a\right] u(p)\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)\,. \tag{11}$$

To check the ward identity, we replace $\epsilon^*_\nu(k_2) \to k_{2\nu}$,

$$i\mathcal{M}^{\mu\nu}_{1,2}\epsilon^*_\mu(k_1)k_{2\nu} = (ig)^2\bar{v}(p_+)\left[\gamma^\mu t^a \frac{i}{\not{p} - \not{k}_2 - m}\not{k}_2 t^b + \not{k}_2 t^b \frac{i}{\not{k}_2 - \not{p}_+ - m}\gamma^\mu t^a\right] u(p)\epsilon^*_\mu(k_1)\,. \tag{12}$$

Using the on-shell condition for the external fermions

$$(\not{p} - m)u(p) = 0\,, \quad \bar{v}(p_+)(-\not{p}_+ - m) = 0\,, \tag{13}$$

we can safely shift k_2 by this quantities so we can get rid of the denominators of the fermion propagators, so we find

$$i\mathcal{M}^{\mu\nu}_{1,2}\epsilon^*_\mu(k_1)k_{2\nu} = (ig)^2\bar{v}(p_+)\left(-i\gamma^\mu[t^a, t^b]\right)u(p)\epsilon^*_\mu(k_1) = -g^2\bar{v}(p_+)\left(f^{abc}t_c\gamma^\mu\right)u(p)\epsilon^*_\mu(k_1)\,. \tag{14}$$

For an abelian theory, $f^{abc} = 0$ and we recover the ward identity. For the non-abelian theory, we must have an additional contribution in order to cancel this non-zero result. This contribution comes from the third diagram

$$\begin{aligned} i\mathcal{M}^{\mu\nu}_3\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2) &= ig\bar{v}(p_+)\gamma_\rho t_c u(p)\frac{-i}{k_3^2}\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2) \\ &\quad \times gf^{abc}\left[\eta^{\mu\nu}(k_2 - k_1)^\rho + \eta^{\nu\rho}(k_3 - k_2)^\mu + \eta^{\rho\mu}(k_1 - k_3)^\nu\right]\,. \end{aligned} \tag{15}$$

We again replace $\epsilon^*_\nu(k_2) \to k_{2\nu}$ and find

$$\begin{aligned} i\mathcal{M}^{\mu\nu}_3\epsilon^*_\mu(k_1)k_{2\nu} &= ig\bar{v}(p_+)\gamma_\rho t_c u(p)\frac{-i}{k_3^2}\epsilon^*_\mu(k_1) \\ &\quad \times gf^{abc}\left[k_2^\mu(k_2 - k_1)^\rho + k_2^\rho(k_3 - k_2)^\mu + \eta^{\rho\mu}(k_1 - k_3)\cdot k_2\right] \\ &= ig^2 f^{abc}\bar{v}(p_+)\gamma_\rho t_c u(p)\frac{-i}{k_3^2}\epsilon^*_\mu(k_1)\left[\eta^{\rho\mu}k_3^2 - k_3^\rho k_3^\mu - \eta^{\rho\mu}k_1^2 + k_1^\rho k_1^\mu\right]\,. \end{aligned} \tag{16}$$

The last three terms vanish. The second term due to on-shell condition of external fermions

$$\bar{v}(p_+)\not{k}_3 u(p) = -\bar{v}(p_+)(\not{p} + \not{p}_+)u(p) = -\bar{v}(p_+)(m - m)u(p) = 0 \tag{17}$$

The third term due to on-shell condition of external massless gauge field

$$k_1^2 = 0\,. \tag{18}$$

Lastly, assuming the other external gauge field is physical (=transversely polarised)

$$\epsilon^*_\mu(k_1)k_1^\mu = 0 \tag{19}$$

the last term vanishes, leaving us with

$$i\mathcal{M}_3^{\mu\nu}\epsilon^*_\mu(k_1)k_{2\nu} = +g^2\bar{v}(p_+)(f^{abc}t_c\gamma^\mu)u(p)\epsilon^*_\mu(k_1)\,, \tag{20}$$

which exactly cancels the contribution from the other two diagrams. Note that the Ward identity relied on the fact the coupling appearing the gauge-fermion vertex is the same coupling appearing in the 3-gauge boson vertex. Conversely, starting from a gauge-invariant theory we find that these couplings must be identical.

$$\text{Gauge invariance} \quad \rightarrow \quad \text{Same } g \text{ in all vertices} \quad \leftarrow \quad \text{Ward identity}\,.$$

But there is something wrong in this derivation!

In order to recover the Ward identity we had to *assume* the other gauge field is physical (=transversely polarized), and our goal was to *prove* that only physical states can be created by a physical process. In order to proceed let us define a complete polarization basis for

$$k^\mu = (k_0, \mathbf{k})\,, \quad k^\mu k_\mu = 0\,. \tag{21}$$

First we define the two (physical) transverse polarizations

$$\epsilon^T_{i\mu} = (0, \hat{n}_i)\,, \quad \hat{n}_i\cdot\mathbf{k} = 0\,, \quad i = 1, 2\,. \tag{22}$$

Additionally we have the longitudinal polarization, proportional to k^μ, and the timelike polarization state, proportional to $\tilde{k}^\mu = (k_0, -\mathbf{k})$

$$\epsilon^\pm_\mu(k) = \frac{1}{\sqrt{2}|\mathbf{k}|}(k_0, \pm\mathbf{k})\,. \tag{23}$$

$\epsilon^+_\mu(k)$ and $\epsilon^-_\mu(k)$ are the forward and backward light-like polarization vectors. The four polarization vectors obey the following orthogonality relations

$$\epsilon^T_i\cdot\epsilon^{*T}_j = -\delta_{ij}\,, \quad \epsilon^+\cdot\epsilon^T_i = \epsilon^-\cdot\epsilon^T_i = 0\,, \tag{24}$$

$$(\epsilon^+)^2 = (\epsilon^-)^2 = 0\,, \quad \epsilon^+\cdot\epsilon^- = 1\,. \tag{25}$$

They satisify the completness relation

$$\eta_{\mu\nu} = \epsilon^-_\mu\epsilon^{+*}_\nu + \epsilon^+_\mu\epsilon^{-*}_\nu - \sum_{i=1,2}\epsilon^T_{i\mu}\epsilon^{T*}_{i\nu}\,. \tag{26}$$

Let us check if this scattering can produce two gauge field in unphysical polarizations, namely $\epsilon^-_\mu(k_1)$ and $\epsilon^+_\mu(k_2)$. We already did most of the work since $\epsilon^+_\mu(k_2) \propto k_{2\mu}$, we can use our result, and account for the different normalization, to find that amplitude to be

$$\begin{aligned} i\mathcal{M}_3^{\mu\nu}\epsilon^{-*}_\mu(k_1)\epsilon^{+*}_\mu(k_2) &= ig^2 f^{abc}\bar{v}(p_+)\gamma_\rho t_c u(p)\frac{-i}{k_3^2}\epsilon^*_\mu(k_1)\frac{1}{\sqrt{2}|\mathbf{k_2}|}\left[k_1^\rho k_1^\mu\right]\,, \\ &= g^2 f^{abc}\bar{v}(p_+)\not{k}_1 t_c u(p)\frac{1}{k_3^2}\frac{|\mathbf{k_1}|}{|\mathbf{k_2}|} \end{aligned} \tag{27}$$

where we used $\tilde{k}^\mu k_\mu = 2|\vec{k}|^2$. So it seems like the amplitude for an unphysical process is non vanishing. Can we just ignore it? When we consider the theory on the classical level (=without quantum corrections, tree-level diagrams), we could. However, a problem appears at the quantum level (=loop diagrams): internal gauge propagators seem to contain unphysical degrees of freedom (remember $\eta_{\mu\nu}$ = sum of all polarizations), whose contribution, in particular to the imaginary part of the diagram, is non-zero as we have seen. In other words, we find that in our naive approach quantum corrections would include contribution from unphysical degrees of freedom, and we can either violate the optical theorem (and therefore unitarity) or allow the production of unphysical states. The suppression of these unphysical degrees of freedom in a non-abelian gauge theory is done with the Faddeev-Popov gauge fixing procedure which introduces additional degrees of freedom known as ghosts. This topic will be covered in the following lectures.

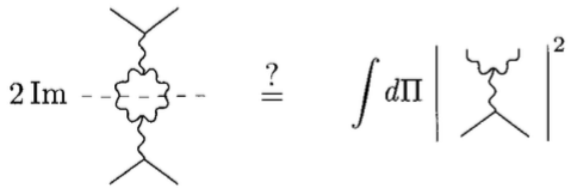

Figure 16.4. A paradox for the optical theorem in gauge theories.

2 Weinberg-Witten theorem

In a paper called "Limits on massless particles" from 1980, Weinberg and Witten prove the following theorems

1. A theory with a Lorentz-covariant conserved charge J_μ cannot contain a massless spin $j > 1/2$ particle which has a non vanishing charge under the conserved charge $Q \equiv \int \mathrm{d}^3x\, J^0$.
2. A theory with a Lorentz-covariant conserved stress-energy tensor $T_{\mu\nu}$ cannot contain a massless spin $j > 1$ particle which has a non vanishing charge under the conserved charge vector $P^\mu \equiv \int \mathrm{d}^3x\, T^{0\mu}$.

Let us consider the basis of our physical states denoted by the two helicity states $|p, \pm j\rangle, |p', \pm j\rangle$. These states satisfy

$$Q\,|p, \pm j\rangle = q\,|p, \pm j\rangle\ , \quad Q\,|p', \pm j\rangle = q\,|p', \pm j\rangle\ , \tag{28}$$

$$P^\mu\,|p, \pm j\rangle = p^\mu\,|p, \pm j\rangle\ , \quad P^\mu\,|p', \pm j\rangle = p'^\mu\,|p', \pm j\rangle\ . \tag{29}$$

The proof is carried out by calculating the matrix elements

$$\lim_{p\to p'} \langle p', \pm j|\, J^\mu\,|p, \pm j\rangle\ , \quad \lim_{p\to p'} \langle p', \pm j|\, T^{\mu\nu}\,|p, \pm j\rangle\ . \tag{30}$$

By assuming charge conservation and Lorentz covariance, we would conclude that having charged massless particles with spins $j > 1/2$ (or $j > 1$ in the second case) leads to a contradiction, rendering the theory inconsistent with Lorentz invariance. First we note that

$$\langle p'|\, Q\,|p\rangle = q\delta^3(p' - p)\,. \tag{31}$$

But also

$$\langle p'|\, Q\,|p\rangle = \int \mathrm{d}^3x\, \langle p'|\, J^0(t,x)\,|p\rangle = \int \mathrm{d}^3x\, \langle p'|\, e^{iP\cdot x} J^0(t,0) e^{-iP\cdot x}\,|p\rangle \tag{32}$$

$$= \int \mathrm{d}^3\, x e^{i(p'-p)x}\, \langle p'|\, J^0(t,0)\,|p\rangle = (2\pi)^3\delta^3(p'-p)\,\langle p'|\, J^0(t,0)\,|p\rangle\ , \tag{33}$$

Therefore

$$\lim_{p'\to p} \langle p'|\, J^0(t,0)\,|p\rangle = \frac{q}{(2\pi)^3}\,. \tag{34}$$

We can generalize this result directly from Lorentz invariance

$$\lim_{p'\to p} \langle p'|\, J^\mu(t,0)\,|p\rangle = \frac{qp^\mu}{E(2\pi)^3} \neq 0\,. \tag{35}$$

Note that the last equation also implies charge conservation. Simlarly we can find

$$\lim_{p'\to p} \langle p'|\, T^{\mu\nu}(t,0)\,|p\rangle = \frac{qp^\mu p^\nu}{E(2\pi)^3} \neq 0\,. \tag{36}$$

Now since p, p' are light-like

$$(p' + p)^2 = 2|\mathbf{p}||\mathbf{p}'|(1 - \cos\theta) \geq 0\,. \tag{37}$$

For $\theta \neq 0$, $p' + p$ is time-like and we can boost to a frame where its spatial component vanishes, so that

$$p = (|\mathbf{p}|, \mathbf{p})\,, \quad p' = (|\mathbf{p}|, -\mathbf{p})\,. \tag{38}$$

First we can find the frame without a spatial component and different energies, and than we boost along the propagation direction and equalize the energies (doppler effect). If we rotate around the propagation axis by an angle ϕ, we know that

$$|p, \pm j\rangle \to e^{\pm i\phi j} |p, \pm j\rangle \;, \quad |p, \pm j\rangle \to e^{\mp i\phi j} |p, \pm j\rangle \tag{39}$$

On the other hand we know how the generator J^μ rotates under Lorentz since it in the fundamental Lorentz representation

$$[L_i, J_0] = 0\,, \quad [L_i, J_j] = i\epsilon_{ijk} J_k\,, \tag{40}$$

Therefore

$$e^{\pm 2i\phi j} \langle p', \pm j| J^\mu(t,0) |p, \pm j\rangle = \Lambda(\phi)^\mu{}_\nu \langle p', \pm j| J^\nu(t,0) |p, \pm j\rangle \tag{41}$$

Since $\Lambda(\phi)^\mu{}_\nu$ can only produce the Fourier components $e^{\pm i\phi}, 1$, for $j > 1/2$ the matrix element must vanish. In the limit $p \to p'$ this leads to a contraction with the non vanishing value we have calculated earlier. We conclude no Lorentz covariant QFT with a conserved current can have charged massless spin-1 (or higher) particles. Similarly,

$$e^{\pm 2i\phi j} \langle p', \pm j| T^{\mu\nu}(t,0) |p, \pm j\rangle = \Lambda(\phi)^\mu{}_\alpha \Lambda(\phi)^\nu{}_\beta \langle p', \pm j| T^{\alpha\beta}(t,0) |p, \pm j\rangle \tag{42}$$

implies that for $j > 1$ the matrix element must vanish. In the limit $p \to p'$ this leads to a contraction with the non vanishing value we have calculated earlier.

Notable exception I : The Gluon

We know we can have massless spin 1 particles in non-abelian theories (e.g the gluon of QCD is massless gauge boson of a gauged, non-abelian $SU(3)$ symmetry, and therefore it is charged under the conserved charged). In the presence of matter, the current is not conserved $\partial_\mu \langle J^\mu_a\rangle \neq 0$, but rather $D_\mu \langle J^\mu_a\rangle = 0$. In the absence of matter, the conserved current of the pure Yang-Mills theory is

$$J^{\mu a} = -F^{\mu\nu}_c f^{cab} A_{\nu b}\,, \tag{43}$$

which by construction is conserved, $\partial_\mu \langle J^{\mu a}\rangle = 0$. However the physical gauge field $A_{\nu a}$ is not Lorentz covariant, since it transforms as

$$A_\mu \to \Lambda^\mu{}_\nu A^\mu + \partial^\mu \Omega(x, \Lambda)\,, \tag{44}$$

which implies of course that $J^{\mu a}$ is not Lorentz covariant,

$$J^{\mu a} \to \Lambda^\mu{}_\nu J^{\nu a} + \dots\,. \tag{45}$$

Therefore the current is not Lorentz covariant and the proof does not apply. In this point one should appreciate the tight connection between the Lorentz and gauge transformations: requiring a Lorentz covariant current is equivalent to requiring that it is gauge invariant! By writing the current in a different form,

$$J_\mu = \partial^\nu F_{\mu\nu}\,, \tag{46}$$

it is clear that the current is not gauge invariant, and therefore not Lorentz covariant. One could restore Lorentz covariance by allowing the field A_μ to have non-physical polarizations, however that would undermine the basis of the proof which relies on the physical basis of the massless particle.

Notable exception II : The Graviton

Gravity can be described as a long range force mediated by a spin-2 particle. The Weinberg-Witten theorem does not apply for the graviton for similar reasons - one cannot construct a Lorentz covariant (or gauge invariant) stress-energy tensor. Writing the metric as linear perturbations around the flat Minkowski metric $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$, one can calculate the stress-energy tensor $T_{\mu\nu}$ as a function of $h_{\mu\nu}(x)$. However, similarly to the gauge field in the non abelian case, $h_{\mu\nu}(x)$ does not transform covariantly under Lorentz.