Relativity, Particles, Fields SS 2017

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Sheet 9: Representations of Lorentz Group, Poincaré Group
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1 Finite Representations of the Lorentz Group

The six generators of the Lorentz group, $J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)$, obey the commutation relations

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}). \tag{1}$$

Define the generators of rotations and boosts as $L^i = (1/2)\epsilon^{ijk} J^{jk}$ and $K^i = J^{i0}$, where $i, j, k = 1, 2, 3$. Infinitesimal Lorentz transformations can then be written as

$$\psi \to (\mathbf{1} - i\boldsymbol{\theta} \cdot \boldsymbol{L} + i\boldsymbol{\eta} \cdot \boldsymbol{K})\psi\,. \tag{2}$$

a) Derive explicitly the commutation relations of the operators L^i and K^i. Then define $\boldsymbol{J}_+ = (\boldsymbol{L} + i\boldsymbol{K})/2$ and $\boldsymbol{J}_- = (\boldsymbol{L} - i\boldsymbol{K})/2$. Show that the components of $\boldsymbol{J}_+$ and $\boldsymbol{J}_-$ separately fulfill the commutation relations of angular momentum and that they commute with each other.

b) Any finite irreducible representation generated by $\boldsymbol{J}_+$ is locally isomorphic to a representation generated by a usual angular momentum, i.e. locally isomorphic to a representation of $SU(2)$. Therefore part **a)** implies that all finite-dimensional representations of the Lorentz group correspond to pairs (j_-, j_+) of integers or half-integers.

Note: Since the $\boldsymbol{J}_+$ are non-hermitian, $\boldsymbol{J}_+^\dagger = \boldsymbol{J}_-$, the *global* structure of the representations is however a non-unitary analytic continuation of the corresponding $SU(2)$ representations. $\boldsymbol{J}_\pm$ generate the group Spin$(1,3) \cong SL(2,\mathbb{C})$, which is in turn the (universal) double cover of the proper orthochronous Lorentz group $SO(1,3)$.

Consider the simplest non-trivial representations. Those are the left- and right-handed Weyl spinors $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. Use the fact that spin-1/2 representations of angular momentum are generated by $\boldsymbol{\sigma}/2$ to show that the Weyl spinors transform as

$$\begin{aligned} \psi^L &\to \Lambda_L \psi^L = \left(\mathbf{1} - (i\boldsymbol{\theta} + \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right) \psi^L\,, \\ \psi^R &\to \Lambda_R \psi^R = \left(\mathbf{1} - (i\boldsymbol{\theta} - \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right) \psi^R\,. \end{aligned} \tag{3}$$

Use $\boldsymbol{\sigma}^* = -\sigma^2 \boldsymbol{\sigma} \sigma^2$ and the explicit form of $\Lambda_{L,R}$ to show that $\sigma^2 \Lambda_L^* \sigma^2 = \Lambda_R$. Show how one can infer from this that if $\psi_L \in (\frac{1}{2}, 0)$, then $\sigma^2 \psi_L^*$ is a right-handed Weyl spinor, i.e. $\sigma^2 \psi_L^* \in (0, \frac{1}{2})$.

c) Prove that if ψ_R and ξ_R are right-handed Weyl spinors and $\sigma^\mu \equiv (1, \boldsymbol{\sigma})$, then $U^\mu = \xi_R^\dagger \sigma^\mu \psi_R$ is a Lorentz four-vector. Show the same for $V^\mu = \xi_L^\dagger \bar{\sigma}^\mu \psi_L$, where ψ_L and ξ_L are left-handed Weyl spinors and $\bar{\sigma}^\mu \equiv (1, -\boldsymbol{\sigma})$.

d) Verify explicitly that for $\Lambda_L = \exp(-i\theta \boldsymbol{n} \cdot \boldsymbol{\sigma}/2)$, $L(\Lambda_L)$ is a rotation by the angle θ around $\boldsymbol{n}$, where L follows from $V^\mu \to V'^\mu = L^\mu_{\ \nu} V^\nu$. Finally, show also that for $\Lambda_L = \exp(-\eta \boldsymbol{n} \cdot \boldsymbol{\sigma}/2)$, $L(\Lambda_L)$ is a boost of rapidity η (i.e. with boost parameters $\beta = \tanh \eta, \gamma = \cosh \eta$) in the direction $\boldsymbol{n}$.

2 Poincaré Group

In the Poincaré group, spacetime translations are added to the set of (homogeneous) Lorentz transformations,

$$x^\mu \to \Lambda^\mu_{\ \nu} x^\nu + a^\mu\,, \qquad a^\mu \in \mathbb{R}^4. \tag{4}$$

Hence a new generator P^μ has to be added to the Lorentz algebra in Eq. (1), to form the Poincaré algebra. The following relations are added:

$$[P^\mu, P^\nu] = 0\,, \qquad [P^\mu, J^{\rho\sigma}] = i(g^{\mu\rho} P^\sigma - g^{\mu\sigma} P^\rho). \tag{5}$$

a) Consider $P^2 = P_\mu P^\mu$ and $W^2 = W_\mu W^\mu$, where $W_\mu \equiv (1/2)\epsilon_{\mu\nu\sigma\rho} P^\nu J^{\sigma\rho}$ is the Pauli-Lubanski polarization vector. Using the commutation relations for the generators of the Poincaré group, show that P^2 and W^2 are Casimir operators for this algebra, i.e. they commute with both $J_{\mu\nu}$ and P_μ.

b) Since the Poincaré group has rank 2, P^2 and W^2 are its only Casimir operators. Therefore a massive state can be labelled by two numbers, its mass and spin. Show that for a massless particle, P^μ and W^μ must be proportional to each other, and use this to conclude that a massless state can be labeled by only one number (called *helicity*).

c) Show that $J^{\mu\nu} J_{\mu\nu}$ is a Casimir of the Lorentz group, but it is not a Casimir of the Poincaré group.