Relativity, Particles, Fields SS 2017

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https://www.t75.ph.tum.de/teaching/ss17-relativity-particles-fields/

Sheet 8: Muon Decay, Equations of motion (27.6.2017)

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1 Muon decay

Consider the process of *muon decay*, $\mu^- \to e^- \bar{\nu}_e \nu_\mu$. The amplitude squared, summed over the spin states of the decay products and averaged over the spin states of the initial muon, is

$$|\mathcal{A}_{fi}|^2 = 64\, G_F^2 (k_1 \cdot k_2')(k_1' \cdot k_3'), \tag{1}$$

where G_F is the *Fermi constant*, k_1 is the four-momentum of the muon, and $k'_{1,2,3}$ are the four-momenta of the $\bar{\nu}_e, \nu_\mu$ and e^-, respectively. In the rest frame of the muon, its decay rate is therefore

$$\Gamma = \frac{32\, G_F^2}{m} \int (k_1 \cdot k_2')(k_1' \cdot k_3') d\Pi_3(k_1), \tag{2}$$

where

$$d\Pi_n(k) \equiv (2\pi)^4 \delta^{(4)}\Big(k - \sum_{j=1}^n k_j'\Big) \prod_{j=1}^n \frac{d^3 k_j'}{(2\pi)^3} \frac{1}{2E_{k_j'}}\,, \tag{3}$$

while $k_1^\mu = (m, \mathbf{0})$ with m the muon mass. All the final state particles can be taken massless.
In this problem we will evaluate Γ through the following analysis:

a) Show that

$$\Gamma = \frac{32\, G_F^2}{m} \int \frac{d^3 k_3'}{(2\pi)^3} \frac{1}{2E_{k_3'}} k_{1\mu} k_{3\nu}' \int k_2'^{\mu} k_1'^{\nu} d\Pi_2(k_1 - k_3'). \tag{4}$$

b) Use Lorentz invariance to argue that

$$\int k_2'^{\mu} k_1'^{\nu} d\Pi_2(q) = A q^2 g^{\mu\nu} + B q^\mu q^\nu\,, \tag{5}$$

where A and B are numerical constants.

c) Show that

$$\int d\Pi_2(q) = \frac{1}{8\pi}\,. \tag{6}$$

Then, by contracting both sides of Eq. (5) with $g_{\mu\nu}$ and $q_\mu q_\nu$ and using Eq. (6), evaluate A and B.

d) Plug the results obtained in **b)** and **c)** into Eq. (4) and compute $d\Gamma/dE_e$, where $E_e \equiv E_{k_3'}$ is the electron energy. Note that the maximum value of E_e is reached when the electron is emitted in one direction, and the two neutrinos in the opposite direction. What is this maximum value?

e) Perform the integral over E_e to obtain the muon decay rate Γ. Using the measured values of the lifetime of the muon, 2.197 μs, and of the muon mass, 105.66 MeV, determine the value of G_F in GeV^{-2}.

f) Define the *energy spectrum* of the electron as $P(E_e) \equiv \Gamma^{-1} d\Gamma/dE_e$. Note that $P(E_e)dE_e$ is the probability for the electron to be emitted with energy between E_e and $E_e + dE_e$. Draw a graph of $P(E_e)$ versus E_e/m.

2 Equations of motion in ϕ^4 theory

Consider the $\lambda\phi^4$ theory,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 .$$

Prove the equations-of-motion identity

$$(\Box_x + m^2)\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = -\frac{\lambda}{3!}\langle 0|T\{\phi(x)^3\phi(y)\}|0\rangle - i\delta^{(4)}(x-y), \tag{7}$$

where ϕ indicates the field in Heisenberg (and not interaction) picture,

a) At $O(\lambda^0)$ and $O(\lambda^1)$;

b*) At $O(\lambda^2)$;

by explicitly computing the left- and right-side at each order in λ, with the help of the Feynman rules.

Hint: You can assume that the time-evolution operator U acts on $|0\rangle$ as $U|0\rangle \to |0\rangle$. Then, you should obtain for the vacuum expectation value (VEV) that appears on the LHS of Eq. (7) an expression that suggests the following physical interpretation: assuming for example $x^0 > y^0$, one particle is created at the spacetime point y, it "propagates" (at appropriate order in λ) until point x, and here it is annihilated. A similar interpretation holds for the first term on the RHS. Use this picture to draw the relevant Feynman diagrams in each case.
Note: To turn the above intuitive picture into a rigorous one, we need to replace in Eq. (7) the ground state of the free theory, $|0\rangle$, with the ground state of the interacting theory, $|\Omega\rangle$. Then one can prove that to compute correlation functions like $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$, only *connected* diagrams need to be considered, i.e. diagrams where every line is connected to an external point. See later lectures and, for reference, the discussion on pages 82-99 of Peskin and Schroeder.

Note 2: If you followed the discussion in the lecture on 29.6, you can alternatively calculate

$$(\Box_x + m^2)\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = -\frac{\lambda}{3!}\langle\Omega|T\{\phi(x)^3\phi(y)\}|\Omega\rangle - i\delta^{(4)}(x-y), \tag{8}$$

and ignore the hint and note above.