Relativity, Particles, Fields SS 2017

Prof. Andreas Weiler (TUM), Dr. Ennio Salvioni (TUM)
https://www.t75.ph.tum.de/teaching/ss17-relativity-particles-fields/

Sheet 6: Classical Source, Wick's Theorem (13.6.2017)

---

1 Particle Creation by a Classical Source

In this exercise we return to the particle creation by a classical source. Recall from Sec. 5.8.2 of the script that this process can be described by the Hamiltonian

$$H = H_0 - \int d^3x \, J(x)\phi(x), \tag{1}$$

where H_0 is the free Klein-Gordon Hamiltonian, $\phi(x)$ is the Klein-Gordon field, and $J(x)$ is the classical, external source (a c-number scalar function). We found that, if the system is in the vacuum state before the source is turned on, the source will create a mean number of particles

$$\langle N \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |\widetilde{J}(p)|^2 \,, \tag{2}$$

where $\widetilde{J}$ is the Fourier transform of J. In this problem we will verify that statement, and extract more detailed information, by using a perturbative expansion in the strength of the source.

a) Show that the probability that the source creates *no* particles is given by

$$P(0) = \left| \langle 0| T \Big\{ \exp \left[i \int d^4x J(x) \phi_I(x) \right] \Big\} |0\rangle \right|^2 . \tag{3}$$

b) Evaluate the term in $P(0)$ of order J^2, and show that $P(0) = 1 - \lambda + O(J^4)$, where $\lambda = \langle N \rangle$ equals the expression given in Eq. (2).

c) Represent the term computed in b) as a Feynman diagram. Now represent the whole perturbation series for $P(0)$ in terms of Feynman diagrams. Show that this series exponentiates, so that it can be summed exactly: $P(0) = e^{-\lambda}$.
Hint: the term computed in b) can be represented as a propagator, and therefore the whole perturbation series for $P(0)$ can be written as an infinite sum over non-interacting propagators, with appropriate symmetry factors.

d) Working at order J^2, compute the probability that the source creates one particle of momentum $\mathbf{k}$. Then integrate this probability over $\mathbf{k}$ to show that the probability to create one particle with *any* momentum is, at $O(J^2)$, $P(1) = \lambda$. Finally, using a representation similar to that employed in point c), show that the all-orders expression of this probability is $P(1) = \lambda e^{-\lambda}$.

e) The all-orders results obtained above for $n = 0$ and $n = 1$ can be generalized to show that the probability to create n particles is

$$P(n) = \frac{\lambda^n e^{-\lambda}}{n!} \,, \tag{4}$$

i.e. it follows the Poisson distribution. Accepting this fact, prove the following properties of $P(n)$:

$$\sum_{n=0}^{\infty} P(n) = 1 \,, \qquad \langle n \rangle = \sum_{n=0}^{\infty} n \, P(n) = \lambda \,. \tag{5}$$

The first property ensures that the $P(n)$ are correctly normalized probabilities, while the second confirms the interpretation of Eq. (2) as the mean number of particles created by the source J.

2 Applications of Wick's Theorem

Working in the $\lambda\phi^4$ theory,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\,\phi^4$$

(all expressions are in the interaction picture, $\phi(x) = \phi_I(x)$), use Wick's theorem to evaluate the following quantities:

a) $\langle 0|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|0\rangle$,

b) $\langle 0|T\{\phi(x_1)\phi(x_2)\int d^4x_3\,\frac{\lambda}{4!}\,\phi^4(x_3)\}|0\rangle$,

c) $\langle 0|T\{\phi^4(x_1)\phi^4(x_2)\}|0\rangle$,

by expressing them in terms of Feynman propagators. In each case, also draw the Feynman diagrams representing the result.