Relativity, Particles, Fields SS 2017

Prof. Andreas Weiler (TUM), Dr. Ennio Salvioni (TUM)

https://www.t75.ph.tum.de/teaching/ss17-relativity-particles-fields/

Sheet 5: Canonical quantization (30.5.2017)

---

1 The charge of a state

Consider a complex scalar field operator ψ. Assume that $|\alpha\rangle$ is an eigenstate of the charge operator Q with eigenvalue q. Prove that the state obtained by applying the operator $\psi^\dagger$, i.e. $\psi^\dagger|\alpha\rangle$, is also an eigenstate of Q. What is the corresponding eigenvalue?

2 Fluctuations in two dimensions

Consider a massless real scalar field ϕ in two-dimensional spacetime.

a) Show that given two space-time points x and y, the fluctuations take the following form

$$\langle 0|\phi(x)\phi(y)|0\rangle = \frac{1}{4\pi}\int_0^\infty \frac{dk}{k}\left(e^{ik(\tau-r)} + e^{ik(\tau+r)}\right) \tag{1}$$

where $\tau \equiv y^0 - x^0$ and $r \equiv y^1 - x^1$.

b) Show that Eq. (1) can be written in the explicit form

$$\langle 0|\phi(x)\phi(y)|0\rangle = -\frac{1}{4\pi}\log\frac{(x-y)^2}{\mu^2}\,, \tag{2}$$

where μ is an integration constant which has the dimension of length.
Hint: a convenient way to proceed is to differentiate the RHS of Eq. (1) with respect to τ. The resulting integral can be regularized by multiplying the integrand times $e^{-\epsilon k}$ with $\epsilon > 0$, and eventually taking the limit $\epsilon \to 0$.

3 Propagator in position space

Consider a scalar field ϕ with mass m, whose Feynman propagator is

$$\Delta_F(x) = \langle 0|T\phi(x)\phi(0)|0\rangle\,. \tag{3}$$

a) Show that $\langle 0|\phi(x)|\mathbf{k}\rangle = e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{r}}$, where $|\mathbf{k}\rangle$ is a one-particle state and $x^\mu = (t, \vec{r})$. Then use this fact to prove that the propagator can be written in the integral form

$$\Delta_F(x) = \frac{1}{4\pi^2}\int_0^\infty dk \frac{k^2}{\omega_k}\frac{\sin kr}{kr}e^{-i\omega_k|t|}\,, \tag{4}$$

with $r = |\vec{r}|$.

b) By Lorentz invariance, Δ_F can depend only on $x^2 = t^2 - r^2$. Assuming a space-like separation $x^2 < 0$, set $t = 0$ and plot: 1) $4\pi^2 r^2 \Delta_F(r)$ in the range $0 < r < 2$, for $m = 1$ and $m = 3$. Observe the $r \to 0$ limit. Does it depend on m? 2) $4\sqrt{2}\,\pi^{3/2}\frac{r^{3/2}}{\sqrt{m}}\Delta_F(r)$ again in the range $0 < r < 2$, for $m = 2$ and $m = 3$. At large r, this tends to an elementary function of mr, $f(mr)$. Determine f and conclude that the typical correlation length is roughly $\Delta r \sim 1/m$.

c) Now set $m = 0$, and prove that in this case the propagator takes the general form

$$\Delta_F(x)_{m=0} = -\frac{i}{4\pi}\delta(x^2) - \frac{1}{4\pi^2}\mathrm{P}\frac{1}{x^2}\,, \tag{5}$$

where P denotes the principal value prescription. To this end, recall that

$$\frac{1}{x \pm i\epsilon} = \mathrm{P}\frac{1}{x} \mp i\pi\delta(x) \tag{6}$$

where $\epsilon > 0$.