Relativity, Particles, Fields SS 2017

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https://www.t75.ph.tum.de/teaching/ss17-relativity-particles-fields/

Sheet 4: Dilatation, Casimir effect and regulators (23.5.2017)

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1 Casimir Effect

In this problem we will explore the Casimir effect[1], which is a force between conducting plates that is seemingly caused by the change in vacuum energy that results from the boundary conditions imposed by the plates. Consider the modes of a free, massless, scalar field in one spatial dimension, confined to a box of length L. We impose the boundary condition that $\phi(x) = 0$ at $x = 0$ and $x = L$, so the allowed terms in the Fourier expansion are $\sin(k_n x)$, where $k_n = n\pi/L$ (with $n = 1, 2, \ldots$). The vacuum energy density will be infinite, as in three-dimensional field theories.

a) Introduce a cut-off factor replacing $\omega_n \to \omega_n e^{-\omega_n/\omega_C}$ to render the vacuum energy finite (After computing the relevant energy difference, we will be able to take the limit $\omega_C \to \infty$). Show that the zero-point energy inside the box is then

$$E_0(L) = \frac{1}{2}\sum_{n=1}^{\infty} \hbar\omega_n e^{-\omega_n/\omega_C}, \quad \text{with} \quad \omega_n = \pi n/L \tag{1}$$

and that

$$E_0(L) = \frac{\pi}{8L}\frac{1}{\sinh^2(\frac{\pi}{2\omega_C L})} \to \frac{L\omega_C^2}{2\pi} - \frac{\pi}{24L} + O\left(\frac{1}{\omega_C^2}\right). \tag{2}$$

Hint: Use that $\sum_n n e^{-\alpha n}$ can be written as the derivative of a geometric series.

b) Now insert two hard-wall partitions in the box centered about the midpoint and separated by distance a. Show that the total zero-point energy $E_{\text{total}}(a)$ becomes

$$E_{\text{total}}(a) = E_0(a) + 2\,E_0(\,(L-a)/2\,). \tag{3}$$

Calculate the force between the partitions using $F = -\frac{\partial E_{\text{total}}(a)}{\partial a}$. Show that in the limit $\omega_C \to \infty$ and $L \to \infty$, the force is

$$F = -\frac{\hbar c \pi}{24 a^2}\,. \tag{4}$$

c) Repeat the calculation with a ζ-function regulator,

$$E_0(L) = \frac{1}{2}\sum_{n=1}^{\infty} \hbar\omega_n \left(\frac{\omega_n}{\mu}\right)^{-s} \tag{5}$$

where we take $s \to 0$ (instead of $\omega_C \to \infty$). Do you get the same results?
Hint: Use $\sum_n n^{1-s} = \zeta(s-1) = -\frac{1}{12} - 0.165\,s + \ldots$.

[1] It has been conclusively observed only very recently: S. K. Lamoreaux, Phys. Rev. Lett. **78**, 5 (1997).

d) Show the independence on the regulator[2] using a generic function $f(x)$,

$$E_0(a) = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{n}{a}f\left(\frac{n}{a\Lambda}\right). \tag{6}$$

Show that this regulator will give the same Casimir force if

$$\lim_{x\to\infty} x\frac{d^j f(x)}{dx^j} = 0,\ \forall j \quad \text{and} \quad f(0) = 1\,. \tag{7}$$

Show that the above regulators satisfy this criterium. Explain why the Casmir force is therefore an infrared effect.

e*) Can you find a physical effect that would be equivalent to $f(x)$ dying off at high energies (or equivalent to the UV modes going right through the plates)? Think of an electro-magnetic field between two metallic plates. When is the system then in the Casimir regime? How can the result for $F(a)$ just depend on fundamental constants $\hbar, c$ and the distance a (and not on $\alpha_{em}, m_e, \ldots$ which it should if it has to do with QED interactions)? Can we conclude that the concept of zero point fluctuations is a heuristic and calculational aid in the description of the Casimir effect, but not a necessity?[3]

2 Dilatation transformations

A class of interesting theories are invariant under the scaling of all lengths by

$$x^\mu \to (x')^\mu = \lambda x^\mu \tag{8}$$

and

$$\phi(x) \to \phi'(x) = \lambda^{-D}\phi(\lambda^{-1}x) \tag{9}$$

Here D is called the scaling dimension of the field. Consider the action for a real scalar field given by

$$S = \int d^4x\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - g\phi^p\right). \tag{10}$$

a) Find the scaling dimension D such that the derivative terms remain invariant. For what values of m and p is the scaling in Eqs. (8) and (9) a symmetry of the theory?

b) How do these conclusions change for a scalar field living in an $(n+1)$-dimensional spacetime instead of a $3+1$-dimensional spacetime?

c) In $3+1$ dimensions, use Noether's theorem to construct the conserved current D^μ associated to scaling invariance.

[2] H.B.G. Casimir, Proc. Kon. Ned. Akad. Wetenschap **B51**, 793 (1948).

[3] See: R. L. Jaffe, The Casimir Effect and the Quantum Vacuum, Phys. Rev. D **72** 021301 (2005).