Relativity, Particles, Fields SS 2017

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https://www.t75.ph.tum.de/teaching/ss17-relativity-particles-fields/

Sheet 3: Classical Field Theory, Canonical Quantization (16.5.2017)

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1 Noether's theorem

Consider the infinitesimal form of the Lorentz transformation, $x^\mu \to x^\mu + \omega^\mu{}_\nu x^\nu$, which we discussed on Sheet **2** (exercise 2).

a) Using Noether's theorem deduce the existence of the conserved current

$$j^\mu = -\omega_{\rho\nu} T^{\mu\rho} x^\nu \,. \tag{1}$$

The three conserved charges arising from spatial rotational invariance define the total angular momentum of the field. Show that these charges are given by

$$Q_i = \epsilon_{ijk} \int d^3x (x^j T^{0k} - x^k T^{0j}). \tag{2}$$

Derive the conserved charges arising from invariance under Lorentz boosts. Show that they imply

$$\frac{d}{dt} \int d^3x (x^i T^{00}) = \text{constant}, \tag{3}$$

and interpret this equation.

b) You have just shown that the classical angular momentum of the field is given by Eq. (2). Write down the explicit form of the angular momentum for a free real scalar field with Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2)$. Show that, after normal ordering, the quantum operator Q_i can be written as

$$Q_i = -i\, \epsilon_{ijk} \int \frac{d^3p}{(2\pi)^3} a^\dagger_{\vec{p}} \left(p^j \frac{\partial}{\partial p_k} - p^k \frac{\partial}{\partial p_j} \right) a_{\vec{p}} \,. \tag{4}$$

Hence confirm that the quanta of the scalar field have spin zero (i.e. a stationary one-particle state $|\vec{p} = 0\rangle$ has zero angular momentum).

2 Stress-energy tensor for electromagnetism

Maxwell's Lagrangian for the electromagnetic field is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ is the 4-vector potential.

a) Show that $\mathcal{L}$ is invariant under gauge transformations $A_\mu \to A_\mu + \partial_\mu \xi$, where $\xi = \xi(x)$ is a scalar field with arbitrary (differentiable) dependence on x.
Then prove that with the identifications $F^{0i} = -E^i$ and $F^{ij} = -\epsilon^{ijk} B^k$ (where E denotes the electric field and B the magnetic field), the Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2).$$

b) Use Noether's theorem, and the spacetime translational invariance of the action, to construct the energy-momentum tensor $T^{\mu\nu}_{\text{Noether}}$ for the electromagnetic field, and show that it is neither symmetric nor gauge invariant.

c) The properly symmetric – and also gauge invariant – stress-energy tensor for the free electromagnetism is

$$T^{\mu\nu}_{\text{EM}} = \; -F^{\mu\lambda}F^{\nu}_{\;\;\lambda} + \tfrac{1}{4}\, g^{\mu\nu}\, F_{\kappa\lambda}F^{\kappa\lambda}\,. \tag{5}$$

Show that this expression indeed has the form $T^{\mu\nu}_{\text{EM}} = T^{\mu\nu}_{\text{Noether}} + \partial_\lambda \mathcal{K}^{[\lambda\mu]\nu}$ for some $\mathcal{K}^{[\lambda\mu]\nu}$, where $[\ldots]$ denotes antisymmetrization.

d) Write down the components of the stress-energy tensor $T^{\mu\nu}_{\text{EM}}$ in non-relativistic notations and make sure you have the familiar electromagnetic energy density, momentum density and pressure.

3 General relativity: perihelion shift of Mercury

In this problem, you will calculate the perihelion shift of Mercury simply by dimensional analysis using the non-linear equations of motions of Einstein gravity as a classical field theory.

a) The interactions of Einstein gravity include the terms

$$\mathcal{L} = M_{\text{Pl}}^2 \left(-\frac{1}{2} h_{\mu\nu} \Box h_{\mu\nu} + (\partial_\alpha h_{\mu\nu})(\partial_\beta h_{\mu\alpha}) h_{\nu\beta} + \ldots \right) - h_{\mu\nu} T_{\mu\nu}\,, \tag{6}$$

where $M_{\text{Pl}} = 1/\sqrt{G_N}$ is the Planck scale. Rescaling h, and dropping indices and numbers of $O(1)$, this simplifies to the Lagrangian of a real scalar field theory for $h(x)$ with derivative interactions

$$\mathcal{L} = -\frac{1}{2} h \Box h + (M_{\text{Pl}})^a h^2 \Box h - (M_{\text{Pl}})^b h T\,. \tag{7}$$

What are a and b (i.e. what are the dimensions of these terms)?

b) The equations of motion following from the Lagrangian in Eq. (7) are (roughly)

$$\Box h = (M_{\text{Pl}})^a \Box (h^2) - (M_{\text{Pl}})^b T\,. \tag{8}$$

For a point source $T = 4\pi m \delta^{(3)}(x)$, solve Eq. (8) for h to *second* order in the source T (or equivalently to third order in M_{Pl}^{-1}). You may use the Coulomb solution that we already derived in the lecture.

c) To first order, h is just the Newtonian potential. This causes Mercury to orbit. What is Mercury's orbital frequency, $\omega = 2\pi/T$? How does it depend on m_{Mercury}, m_{Sun}, M_{Pl} and the distance R between Mercury and the Sun?

d) To second order, there is a correction that causes a small shift to Mercury's orbit. Estimate the order of magnitude of the correction to ω in arcseconds/century using your second-order solution.

e) If you derive Eq. (8) from Eq. (7), what additional terms do you get? Why is it OK to use Eq. (8) without these terms?