Sheet 2: Relativity, Classical Field Theory (9.5.2017)

1 The Greisen-Zatsepin-Kuzmin (GZK) cutoff

In complete analogy to other composite objects, such as atoms or nuclei, protons can be excited when bombarded with photons. The lowest excited state of the proton is called Δ^+ and has a mass $m_{\Delta^+} = 1232$ MeV.

a) Consider a gas of photons at temperature $T \ll m_p$. Assume for simplicity that all photons have an energy equal to the average thermal energy and calculate the corresponding energy of a proton going through that gas necessary to produce a Δ^+.

b) The Δ^+ eventually decays via $\Delta^+ \to p\pi^0$, with $m_{\pi^0} = 134$ MeV. Calculate the average fraction of the proton energy that is lost per collision as the proton goes through the photon gas.
Hint: You can assume that the decay of the Δ^+ is isotropic in its rest frame.

c) The Universe is filled by a gas of photons, the cosmic microwave background (CMB), with temperature $T = 2.7$ K. Protons propagating through the CMB undergo the above-mentioned process and lose energy. As a consequence, above a certain energy the number of protons reaching the Earth drops dramatically. Calculate this energy, which is known as the Greisen-Zatsepin-Kuzmin (GZK) cutoff.

2 Lorentz transformations

A Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ is such that it preserves the Minkowski metric $\eta_{\mu\nu}$, meaning that $\eta_{\mu\nu} x^\mu x^\nu = \eta_{\mu\nu} x'^\mu x'^\nu$ for all x.

a) Show that this implies that $\eta_{\mu\nu} = \eta_{\sigma\tau} \Lambda^\sigma{}_\mu \Lambda^\tau{}_\nu$. Then, use this result to show that an infinitesimal transformation of the form $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$ is a Lorentz transformation when $\omega^{\mu\nu}$ is antisymmetric, i.e. $\omega^{\mu\nu} = -\omega^{\nu\mu}$.

b) Write down the matrix form for $\omega^\mu{}_\nu$ that corresponds to a rotation by an infinitesimal angle θ about the x^3-axis. Do the same for a boost along the x^1-axis by an infinitesimal velocity v.
Hint: See lecture notes.

c) Consider the infinitesimal form of the Lorentz transformation derived in **a)**: $x^\mu \to x^\mu + \omega^\mu{}_\nu x^\nu$. Show that a scalar field transforms as

$$\phi(x) \to \phi'(x) = \phi(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi(x) \tag{1}$$

and hence show that the variation of the Lagrangian density is a total derivative,

$$\delta \mathcal{L} = -\partial_\mu (\omega^\mu{}_\nu x^\nu \mathcal{L}). \tag{2}$$

3 Functional derivative

For a functional $F[\phi]$ acting on a function $\phi(x)$, the functional derivative $\delta F[\phi]/\delta\phi(x)$ is defined via

$$F[\phi+\eta] = F[\phi] + \int dx' \frac{\delta F[\phi]}{\delta\phi(x')}\,\eta(x') + \ldots$$

where $\eta(x)$ is an infinitesimal function, and the dots stand for terms of higher order in η. For example, for the functional $F_x[\phi] = \phi(x)$ one finds

$$\frac{\delta\phi(x)}{\delta\phi(x')} = \delta(x-x'), \tag{3}$$

and for the functional $Q[\phi] = \int dx(\partial_x\phi(x))^2$ we have

$$\frac{\delta Q[\phi]}{\delta\phi(x)} = -2\partial_x^2\phi(x)\,, \tag{4}$$

where we have assumed that the functions vanish sufficiently fast at the boundary, so that surface terms vanish.

a) Derive from the definition above the product rule for the functional derivative,

$$\frac{\delta(F[\phi]G[\phi])}{\delta\phi(x)} = \frac{\delta F[\phi]}{\delta\phi(x)}G[\phi] + F[\phi]\frac{\delta G[\phi]}{\delta\phi(x)}\,. \tag{5}$$

b) Prove also the chain rule

$$\frac{\delta F[G[\phi]]}{\delta\phi(x)} = \int dy \frac{\delta F[G]}{\delta G(y)}\frac{\delta G[\phi]}{\delta\phi(x)}\,, \tag{6}$$

where $G : \phi(x) \to G(y)$ associates a function $G(y)$ to a function $\phi(x)$: therefore, for fixed y, G is a functional $G[\phi]$, and for fixed ϕ, G is a function $G(y)$, for example

$$G(y) = \int dx K(x-y)\phi(x)\,, \qquad F[G] = \int dy (G(y))^2\,. \tag{7}$$

c) Show that the Euler-Lagrange equations are equivalent to the functional equations for the action

$$\frac{\delta S[\phi]}{\delta\phi(x)} = 0\,. \tag{8}$$

d) Use functional derivatives to obtain the equation of motion for the field ϕ from

$$S = \int d^4x\big[\dot\phi^2(x) - a\phi^2(x) - b\phi^4(x) - c^2(\nabla\phi(x))^2\big]. \tag{9}$$