Relativity, Particles, Fields

SS 2017

Prof. Andreas Weiler (TUM), Dr. Ennio Salvioni (TUM)

https://www.t75.ph.tum.de/teaching/ss17-relativity-particles-fields/

Sheet 1: Relativity (2.5.2017)

1 Discrete chain (lecture)

The Hamiltonian of the quantum discrete chain is

$$H(\hat{p},\hat{q}) = \sum_{j=1}^{N}\left[\frac{\hat{p}_j^2}{2m} + \frac{\kappa}{2}(\hat{q}_j - \hat{q}_{j+1})^2\right]$$

with $p_j = m\dot{q}_j$ and

$$[\hat{p}_j, \hat{q}_k] = -i\hbar\delta_{jk} \tag{1}$$

Periodic boundary conditions give the normal modes

$$\hat{q}_j(t) = \frac{1}{\sqrt{N}}\sum_{n=-N/2}^{N/2}\hat{Q}_n\, e^{i2\pi nj/N}, \qquad \hat{p}_j(t) = \frac{1}{\sqrt{N}}\sum_{n=-N/2}^{N/2}\hat{P}_n\, e^{i2\pi nj/N}\,. \tag{2}$$

a) Show that the operators $\hat{P}_n$ and $\hat{Q}_n$ satisfy

$$[\hat{P}_n^\dagger, \hat{Q}_m] = -i\delta_{nm}, \qquad \hat{P}_n^\dagger = \hat{P}_{-n}, \qquad \hat{Q}_n^\dagger = \hat{Q}_{-n}\,. \tag{3}$$

b) Show that the substitution of Eq. (2) into H gives

$$H = \sum_{n=1}^{N}\left[\frac{1}{2m}\hat{P}_n^\dagger\hat{P}_n + \frac{1}{2}m\omega_n^2\,\hat{Q}_n^\dagger\hat{Q}_n\right] \tag{4}$$

with eigen-frequencies $\omega_n^2 = \frac{4\kappa}{m}\sin^2\left(\pi\frac{n}{N}\right)$.

c) Show that with

$$\hat{a}_k \equiv \sqrt{\frac{m\omega_k}{2}}\left(\hat{Q}_k + \frac{i}{m\omega_k}\hat{P}_k\right), \qquad \hat{a}_k^\dagger \equiv \sqrt{\frac{m\omega_k}{2}}\left(\hat{Q}_{-k} - \frac{i}{m\omega_k}\hat{P}_{-k}\right) \tag{5}$$

we get the standard form of the harmonic oscillator in terms of creation and annihilation operators

$$H = \sum_{k=-N/2}^{N/2}\omega_k\left(\hat{a}_k^\dagger\hat{a}_k + \frac{1}{2}\right). \tag{6}$$

2 Addition of velocities

Derive the law of addition of velocities in general, when $\vec{u}$ and $\vec{v}$ point in different directions. The law must obey rotational invariance.

3 Ordering of space-time events

Three events A, B, C are seen by observer $\mathcal{O}$ to occur in the order ABC. Another observer, $\mathcal{O}'$, sees the events to occur in the order CBA. Can there always be a third observer who sees the events in the order ACB? Restrict your considerations to one space dimension and support your conclusion by drawing a spacetime diagram.

4 Atmospheric muon lifetime

Muons are unstable subatomic particles with an approximate mass of $m_\mu = 100$ MeV$/c^2$ and a lifetime of $\tau = 1.5\ \mu s$. They can be created when cosmic rays enter the Earth's atmosphere. Suppose that a muon is created at 4 km above sea level with an energy of 1000 MeV with respect to the Earth, taking it 1.5 μs to disintegrate in the muon's rest frame. Will the muon reach the sea level for

a) an observer in the Earth's rest frame?
b) an observer in the muon's rest frame?

5 Length contraction and time dilation

A train of proper length L enters a tunnel of proper length D, travelling at a velocity v.
[**a**] Find the condition between L, D and v, such that the train is totally inside the tunnel at some time for an observer A standing outside the tunnel. What is the corresponding condition for an observer B inside the train?
[**b**] When the back end of the train enters the tunnel, observer B realizes that the exit of the tunnel is closed. Show that the condition on the time observer B will have, in his/her rest frame, to react and send a laser signal from the back end of the train to the exit of the tunnel, in order to make a hole for the train to pass through, is given by

$$\tau_B < \frac{D\sqrt{1-\beta^2} - L(1+\beta)}{v}\,, \tag{7}$$

where $\beta = v/c\,$.