Relativity, Particles, Fields SS 2017

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https://www.t75.ph.tum.de/teaching/ss17-relativity-particles-fields/

Sheet 11: Quantization of Dirac field, QED (18.7.2017)

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1 Bosonic quantization of Dirac field

The purpose of this exercise is to verify that attempting to quantize the Dirac field as a boson leads to irreparable problems. We start from the standard decomposition in the Schroedinger picture,

$$\begin{aligned}\Psi(\boldsymbol{x}) &= \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[b^s_{\boldsymbol{p}} u^s(\boldsymbol{p}) e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + c^{s\dagger}_{\boldsymbol{p}} v^s(\boldsymbol{p}) e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \right], \\ \Psi^\dagger(\boldsymbol{x}) &= \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[b^{s\dagger}_{\boldsymbol{p}} u^s(\boldsymbol{p})^\dagger e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} + c^{s}_{\boldsymbol{p}} v^s(\boldsymbol{p})^\dagger e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \right],\end{aligned} \tag{1}$$

and postulate the commutation relations

$$[b^r_{\boldsymbol{p}}, b^{s\dagger}_{\boldsymbol{q}}] = B(2\pi)^3\delta^{rs}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}), \qquad [c^r_{\boldsymbol{p}}, c^{s\dagger}_{\boldsymbol{q}}] = C(2\pi)^3\delta^{rs}\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q}), \tag{2}$$

where B and C are constants, while all other commutators vanish.

a) Find the values of B and C such that bosonic canonical commutation relations with $\pi_\beta = i\Psi^\dagger_\beta$ are satisfied, i.e.

$$[\Psi_\alpha(\boldsymbol{x}), \Psi^\dagger_\beta(\boldsymbol{y})] = \delta_{\alpha\beta}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}), \qquad [\Psi_\alpha(\boldsymbol{x}), \Psi_\beta(\boldsymbol{y})] = [\Psi^\dagger_\alpha(\boldsymbol{x}), \Psi^\dagger_\beta(\boldsymbol{y})] = 0, \tag{3}$$

where α, β are indices that run over the spinor components.

b) Write down the Hamiltonian density $\mathcal{H}$ that corresponds to the Dirac Lagrangian $\mathcal{L} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi$. Then, using the results of part **a)**, show that after normal ordering the following expression is obtained,

$$H = \int d^3x\, \mathcal{H} = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3} E_p \left(b^{s\dagger}_{\boldsymbol{p}} b^s_{\boldsymbol{p}} - c^{s\dagger}_{\boldsymbol{p}} c^s_{\boldsymbol{p}} \right). \tag{4}$$

Notice the minus sign in front of the second term. This Hamiltonian is not bounded below, since you can lower the energy indefinitely by creating more and more c particles. This is one way to see that a theory of bosonic spin-1/2 particles is ill-defined – a glimpse at the spin-statistics theorem.

2 $e^+e^- \to \mu^+\mu^-$ at lowest order in QED

Consider the QED Lagrangian describing the interactions of the electron and muon with the electromagnetic field,

$$\mathcal{L} = \sum_{i=e,\mu} \bar{\psi}_i(i\not{D} - m_i)\psi_i - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{5}$$

where $D_\mu = \partial_\mu - ieA_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In this problem you will calculate the cross section for $e^+e^- \to \mu^+\mu^-$ scattering at the lowest order in QED.

a) Convince yourself that there is only one Feynman diagram that contributes to the amplitude at tree level. Label p_1 (p_2) the four-momentum of the incoming electron (positron) and k_1 (k_2) the momentum of the outgoing muon (antimuon). Using the following Feynman rules:

$$ie\gamma^\mu \qquad (\bar{\psi}A^\mu\psi \text{ vertex}), \qquad \frac{-ig^{\mu\nu}}{q^2} \qquad (\text{propagator of photon with momentum } q), \tag{6}$$

write down the amplitude $i\mathcal{M}(ee \to \mu\mu)$. Then square it, obtaining

$$|\mathcal{M}|^2 = \frac{e^4}{s^2}\Big(\bar{v}^r(p_2)\gamma^\mu u^s(p_1)\bar{u}^s(p_1)\gamma^\nu v^r(p_2)\Big)\Big(\bar{u}^q(k_1)\gamma_\mu v^p(k_2)\bar{v}^p(k_2)\gamma_\nu u^q(k_1)\Big), \tag{7}$$

where $s = (p_1 + p_2)^2 = (k_1 + k_2)^2$ is one of the Mandelstam variables.

b) Now sum over the spin states of the final muons, and average over the spin states of the initial electrons (explain why this corresponds to the scattering of unpolarized electrons). In practice, this means you should compute $\frac{1}{2}\sum_s \frac{1}{2}\sum_r \sum_{q,p} |\mathcal{M}|^2$, for which you should find (setting $m_e = 0$ henceforth)

$$\overline{|\mathcal{M}|^2} \equiv \frac{1}{4}\sum_{s,r,q,p} |\mathcal{M}|^2 = \frac{e^4}{4s^2}\mathrm{Tr}\big[\not{p}_2\gamma^\mu \not{p}_1\gamma^\nu\big]\,\mathrm{Tr}\big[(\not{k}_1 + m_\mu)\gamma_\mu(\not{k}_2 - m_\mu)\gamma_\nu\big]. \tag{8}$$

c) By computing explicitly the traces, arrive at the expression

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{s^2}\left[(p_1\cdot k_1)(p_2\cdot k_2) + (p_1\cdot k_2)(p_2\cdot k_1) + m_\mu^2(p_1\cdot p_2)\right]. \tag{9}$$

Hint: you will find some of the identities derived in Ex. 3 of Sheet 10 useful.
Now specialize to the center of mass frame, and show that Eq. (9) can be expressed as

$$\overline{|\mathcal{M}|^2} = e^4\left[\left(1 + \frac{4m_\mu^2}{s}\right) + \left(1 - \frac{4m_\mu^2}{s}\right)\cos^2\theta\right], \tag{10}$$

where θ is the angle between the electron momentum $\vec{p}_1$ and the muon momentum $\vec{k}_1$.

d) Finally, calculate the differential scattering cross section

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CoM}} = \frac{1}{2E_{e^-}2E_{e^+}|v_{e^-} - v_{e^+}|}\,\frac{|\vec{k}_1|}{(2\pi)^2 4\sqrt{s}}\,\overline{|\mathcal{M}|^2}, \tag{11}$$

where $|v_{e^-} - v_{e^+}|$ is the relative velocity of the electron and positron. Express the RHS of this equation as a function of the variables s and $\cos\theta$, then perform the integral in $d\Omega$ to obtain the total cross section,

$$\sigma(ee \to \mu\mu) = \frac{4\pi\alpha^2}{3s}\sqrt{1 - \frac{4m_\mu^2}{s}}\left(1 + \frac{2m_\mu^2}{s}\right), \tag{12}$$

where $\alpha \equiv e^2/(4\pi)$ is the fine structure constant.