Relativity, Particles, Fields SS 2017

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https://www.t75.ph.tum.de/teaching/ss17-relativity-particles-fields/

Sheet 10: Dirac Equation, Gamma Matrices (11.7.2017)

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1 Lorentz invariance of the Dirac Equation

Given a set of four $n \times n$ matrices γ^μ that satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,, \tag{1}$$

we can write an n-dimensional representation of the Lorentz algebra,

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]\,. \tag{2}$$

a) Show that the $S^{\mu\nu}$ indeed satisfy the commutation relations for the generators of the Lorentz group,

$$[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}). \tag{3}$$

b) Given an infinitesimal Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ with $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$, define $S(\Lambda) = \exp\left(-i\,\omega_{\mu\nu}S^{\mu\nu}/2\right)$. Then the Dirac spinors transform as $\psi'(x') = S(\Lambda)\psi(x)$. Show that

$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu\,. \tag{4}$$

Then show that the Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$ is Lorentz invariant.

c) Define $\overline{\psi} \equiv \psi^\dagger\gamma^0$ and $\overline{M} \equiv \gamma^0 M^\dagger \gamma^0$, with M an arbitrary 4×4 matrix. Prove that

$$\begin{aligned} \overline{M_1 M_2} &= \overline{M}_2 \overline{M}_1\,, \\ (\overline{\psi}_1 M \psi_2)^* &= \overline{\psi}_2 \overline{M} \psi_1\,, \\ \overline{\gamma^\mu} &= \gamma^\mu\,. \end{aligned} \tag{5}$$

Prove also that $\overline{S}(\Lambda) = \gamma^0 S^\dagger(\Lambda)\gamma^0 = S^{-1}(\Lambda)$, and use this result to find the transformation behavior of $\overline{\psi}(x) = \psi^\dagger(x)\gamma^0$.

2 Representations of Clifford algebra and Dirac Spinors

The Weyl or chiral representation of the Clifford algebra is given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{6}$$

a) Show that the above matrices indeed satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Then find a unitary matrix U such that $(\gamma')^\mu = U\gamma^\mu U^\dagger$, where $(\gamma')^\mu$ form the Dirac representation of the Clifford algebra,

$$(\gamma')^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (\gamma')^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{7}$$

b) The plane-wave solutions of the Dirac equation are $\psi(x) = u^s(p)e^{-ix\cdot p}$ or $\psi(x) = v^s(p)e^{+ix\cdot p}$, where in the chiral representation

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix}, \qquad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar\sigma}\,\eta^s \end{pmatrix}, \tag{8}$$

where $\sigma^\mu = (1,\vec\sigma)$ and $\bar\sigma^\mu = (1,-\vec\sigma)$, while ξ^s $(s = 1,2)$ is a basis of orthonormal two-component spinors, satisfying $(\xi^r)^\dagger\xi^s = \delta^{rs}$, and similarly for η^s. Show that

$$\begin{aligned} u^r(p)^\dagger u^s(p) &= 2E_p\delta^{rs}, \qquad & \bar u^r(p)u^s(p) &= 2m\delta^{rs} \\ v^r(p)^\dagger v^s(p) &= 2E_p\delta^{rs}, \qquad & \bar v^r(p)v^s(p) &= -2m\delta^{rs}, \end{aligned} \tag{9}$$

while the orthonormality conditions are

$$\bar u^s(p)v^r(p) = 0, \qquad u^s(\mathbf{p})^\dagger v^r(-\mathbf{p}) = 0. \tag{10}$$

c) Finally, show that

$$\sum_{s=1}^{2} u^s(p)\bar u^s(p) = \not p + m, \qquad \sum_{s=1}^{2} v^s(p)\bar v^s(p) = \not p - m. \tag{11}$$

3 Useful identities for gamma matrices

Using only the algebra in Eq. (1) (that is, without resorting to a particular representation), and defining the object $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, prove the following results:

a)

1. $\mathrm{Tr}\,\gamma^\mu = 0$,
2. $\mathrm{Tr}\,\gamma^\mu\gamma^\nu = 4\,g^{\mu\nu}$,
3. $\not p_1 \not p_2 = 2p_1\cdot p_2 - \not p_2 \not p_1$,
4. $\gamma_\mu \not p\, \gamma^\mu = -2\not p$ and $\gamma_\mu \not p_1 \not p_2 \gamma^\mu = 4p_1\cdot p_2$,
5. $\mathrm{Tr}(\not p_1 \not p_2 \not p_3 \not p_4) = 4\,[(p_1\cdot p_2)(p_3\cdot p_4) - (p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)]$,
6. $\mathrm{Tr}(\not p_1 \dots \not p_n) = 0$ if n is odd,

b)

1. $(\gamma^5)^2 = 1$ and $\mathrm{Tr}\,\gamma^5 = 0$,
2. $\mathrm{Tr}(\gamma^5 \not p_1 \not p_2) = 0$,
3. $\mathrm{Tr}(\gamma^5 \not p_1 \not p_2 \not p_3 \not p_4) = -4i\,\epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma$,
4. $\{\gamma^\mu, \gamma^5\} = 0$ and $[S^{\mu\nu}, \gamma^5] = 0$.

Hint: Some useful tricks include using the cyclicity of the trace, and inserting $(\gamma^5)^2 = 1$ into a trace.