Quantum Field Theory WS 2018/2019

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http://www.t75.ph.tum.de/teaching/ws18-quantum-field-theory/

Sheet 9: renormalization
(17.12.2018; solution due by 09.01.2019 at 16:00, the parts required for hand-in will be announced on 08.01.2019 at 8:00am; discussed at tutorials of 09.01, 10.01 and 14.01)

1*[1] Feynman parameters

Prove the Feynman parameter identity

$$\frac{1}{P_1^{a_1} P_2^{a_2} \cdots P_n^{a_n}} = \frac{\Gamma(a_1 + \cdots + a_n)}{\Gamma(a_1) \cdots \Gamma(a_n)} \times \int_0^1 dx_1 \cdots dx_n \, \delta(1 - x_1 - \cdots - x_n) \, \frac{x_1^{a_1 - 1} \cdots x_n^{a_n - 1}}{(x_1 P_1 + \cdots + x_n P_n)^{a_1 + \cdots + a_n}} \,. \tag{1}$$

Hint: Make use of the integral $\int_0^\infty dt \, e^{-tP} t^{a-1} = \Gamma(a) P^{-a}$. You may assume that $P_i > 0$, $a_i > 0$.

2* Optical theorem

a) Define the T-operator through $S = 1 + i\,T$, where S is the scattering operator. Derive the identity

$$i(T^\dagger - T) = T^\dagger T \,, \tag{2}$$

and show that it is equivalent to

$$T_{\beta\alpha} - T^*_{\alpha\beta} = i \sum\!\!\!\!\!\!\int_\gamma d\Pi_\gamma (2\pi)^4 \delta^{(4)}(p_\alpha - p_\gamma) T^*_{\gamma\beta} T_{\gamma\alpha} \,, \tag{3}$$

where $\sum\!\!\!\!\!\!\int_\gamma$ denotes a sum/integral over a complete set of (multi)particle states labeled by their three-momenta and possible other quantum numbers.

Apply the above equation to the case where $\alpha = \beta$ is a single-particle state at rest to derive a relation between the imaginary part of the self-energy and the total decay width of the particle. Note that in the general case of an unstable particle, the LSZ formula applies to the on-shell limit $p^2 \to m_*^2$, where $m_*^2 = m^2 - im\Gamma$ is the complex pole location of the two-point function, and this relation defines the physical mass m and width Γ.

b) Assume a field theory of two real scalar fields ϕ, a with Lagrangian

$$\mathcal{L} = \frac{1}{2} \left(\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2 \right) + \frac{1}{2} \partial^\mu a \partial_\mu a - g \phi a^2 \,. \tag{4}$$

Compute the physical mass for both scalar particles at the one-loop order in terms of the bare mass as well as the residues of the two-point function of bare fields (that is, the on-shell wave function renormalization constants) in dimensional regularization. Note that for the ϕ particle the location of the pole becomes complex. What is the physical interpretation of the imaginary part? On a different note, which further terms must be included in the Lagrangian to render it renormalizable?

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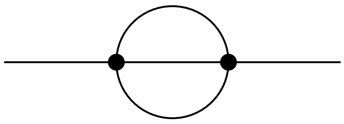
[1] The * means to be handed in.

3 Loops in dim. reg. [discussed at central tutorial of 23.01]

Consider the two-loop contribution to the self-energy of a real massless (i.e. $m = 0$) scalar field in ϕ^4 theory.

a) Usually the result for a two-loop diagram with divergent subdiagrams contains a $1/\epsilon^2$ double pole. Explain why there is only a single $1/\epsilon$ pole for the diagram below, and why there would be a double pole if $m \neq 0$. No explicit calculations are required.



b) Compute (in the massless theory) the complete two-loop self-energy $-i\Pi$ renormalized in the $\overline{\text{MS}}$ scheme, i.e. the self-energy at order λ^2.

Hints: Use that in dimensional regularization

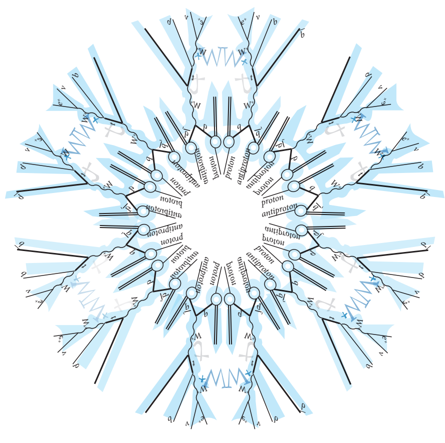
$$\int \frac{d^dk}{(2\pi)^4} \frac{1}{k^2} = 0\,. \tag{5}$$

Why does this equation hold? More generally, why do all diagrams independent of external scales (masses, momenta) vanish in dimensional regularization?

For part **b)** it proves useful to first compute the integral

$$\tilde{\mu}^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} \frac{1}{[-(p+k)^2 - i\eta][-k^2 - i\eta]^a}\,, \tag{6}$$

for arbitrary a and without expanding in ϵ, with the help of Feynman's parametrization.



Feynman diagram snowflake!