Quantum Field Theory WS 2018/2019

Prof. Andreas Weiler (TUM); Dr. Patrick Vaudrevange,
Dr. Ennio Salvioni, Dr. Javi Serra
http://www.t75.ph.tum.de/teaching/ws18-quantum-field-theory/

Sheet 7: more on scattering and particle decay
(05.12.2018; solution due by 12.12 at 16:00, the parts required for hand-in will be announced on 11.12 at 8:00am; discussed at tutorials of 12.12, 13.12 and 17.12)

---

1 Decay rate

In this problem we derive the relation between the S-matrix element and the observable decay rate of a particle.

a) Start from the transition rate for $\alpha \to \beta$,

$$\frac{|\langle \psi_\beta^{\text{out}} | \psi_\alpha^{\text{in}} \rangle|^2}{T}\,, \tag{1}$$

and show, by putting the system in a finite volume V, that it can be written as

$$V(2\pi)^4 \delta^{(4)}(p_\beta - p_\alpha) \prod_i \left(\frac{N_i}{2p_i^0 (2\pi)^3} \right) \prod_j \left(\frac{N_j}{2p_j'^0 (2\pi)^3} \right) |T_{\beta\alpha}|^2\,, \tag{2}$$

where N_p are normalization factors and $T_{\beta\alpha}$ is the T-matrix element in the conventional (continuum) normalization for the states.

b)*[1] Now we need to multiply Eq. (2) by $\sim d^3 p_j'$ for each final state particle, since we can only calculate the transition rate into final states that have momenta in a small interval around $\vec{p}_j'$. Using the completeness relation for the particle states, show that the correct factor is

$$dR_{\beta\alpha} = (\text{Equation 2}) \times \prod_j \frac{d^3 p_j'}{N_j}\,. \tag{3}$$

c) Specialize to the case of a $1 \to n'$ transition, and show that in this case

$$dR_{\beta\alpha} = d\Gamma_{\beta\alpha} = \frac{1}{2p_i^0} |T_{\beta\alpha}|^2 d\Pi_{\text{LIPS}}^{(n')}\,, \tag{4}$$

where p_i^0 is the energy of the decaying particle, and $d\Pi_{\text{LIPS}}^{(n')}$ is the Lorentz-invariant phase space for the n' final-state particles.

d)* Now consider two real scalar fields Φ, ϕ described by the Lagrangian $\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}^i$, where

$$\mathcal{L}_{\text{free}} = \frac{1}{2}(\partial_\mu \Phi \partial^\mu \Phi - M^2 \Phi^2 + \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) \tag{5}$$

and

$$\mathcal{L}_{\text{int}}^1 = -\frac{\mu}{2}\, \Phi \phi\phi\,, \qquad \mathcal{L}_{\text{int}}^2 = \frac{\Phi}{2f}\, \partial_\mu \phi \partial^\mu \phi\,. \tag{6}$$

Assuming $M > 2m$ calculate the leading-order lifetime $\Gamma(\Phi \to \phi\phi)$, both in the case $i = 1$, where $|T_{\beta\alpha}|^2 = \mu^2$, and in the case $i = 2$, where $|T_{\beta\alpha}|^2 = (p_1 \cdot p_2)^2/f^2$, with $p_{1,2}$ the four-momenta of the final-state ϕ particles.

---

[1] The * means to be handed in.

2* $2 \to 2$ scattering

This problem is devoted to the kinematics and differential cross sections of $2 \to 2$ scattering reactions. For a general scattering process of the type $a + b \to c + d$ one defines the Mandelstam variables as

$$s = (p_a + p_b)^2, \qquad t = (p_a - p_c)^2, \qquad u = (p_a - p_d)^2 \tag{7}$$

where p_i are the four-momenta of the particles, whose squared masses are denoted by m_i^2.

a) Show that

$$s + t + u = \sum_{i=a,b,c,d} m_i^2 \,. \tag{8}$$

b) Express the energies of the particles in the center-of-mass (cms) frame as functions of s and the particle masses. Show that for the momenta

$$|\vec{p}_a| = |\vec{p}_b| = \frac{\sqrt{\lambda(s, m_a^2, m_b^2)}}{2\sqrt{s}} \tag{9}$$

(and similarly for $|\vec{p}_c|, |\vec{p}_d|$), where

$$\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz \,. \tag{10}$$

Show also that the flux factor $\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}$ in the cross section formula equals $\sqrt{s}\,|\vec{p}_a|$.

c) Explain why, for fixed cms energy $\sqrt{s}$, the T-matrix element can be considered as a function of the single variable t (or alternatively, the scattering angle θ, where θ is the angle between $\vec{p}_c$ and $\vec{p}_a$ in the cms frame).

This result holds only for particles without spin, as assumed here. Explain why this result cannot be expected to be true in general. Under what conditions *does* it hold for particles with spin?

d) Assuming again that all particles are spin-less, express the differential cross sections $d\sigma/d\Omega$, $d\sigma/d\cos\theta$ and $d\sigma/dt$ in terms of kinematic factors and the T matrix elements ($d\Omega$ denotes the solid angle element). In the case of $d\sigma/dt$, what is the range of allowed values for t?