Quantum Field Theory WS 2018/2019

Prof. Andreas Weiler (TUM); Dr. Patrick Vaudrevange,
Dr. Ennio Salvioni, Dr. Javi Serra
http://www.t75.ph.tum.de/teaching/ws18-quantum-field-theory/

Sheet 6: scattering, Lippmann-Schwinger, and particle decay
(28.11.2018; solution due by 05.12 at 16:00, the parts required for hand-in will be announced on 04.12 at 8:00am; discussed at tutorials of 05.12, 06.12 and 10.12)

---

1 Scattering essentials

a) Prove the relation

$$p_1^0 p_2^0 |\vec{v}_1 - \vec{v}_2| = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} \tag{1}$$

for the head-on collision of two particles with four-momenta $p_{1,2}$, which is needed in the derivation of the formula for the scattering cross section.

b)*[1] Using the Lippmann-Schwinger equation, show that for a wave-packet state

$$e^{i\widetilde{H}_0 t} e^{-iHt} \int d\alpha f(\alpha) |\psi_\alpha^{\text{in/out}}\rangle \quad \overset{t\to\mp\infty}{\longrightarrow} \quad \int d\alpha f(\alpha) |\phi_\alpha\rangle \tag{2}$$

where $|\psi_\alpha^{\text{in/out}}\rangle$ are in- and out-states, and $|\phi_\alpha\rangle$ are free particle states (in the sense discussed in the lecture, i.e. eigenstates of $\widetilde{H}_0$).

2 Scattering in quantum mechanics: Lippmann-Schwinger equation

In this problem we elucidate the relation of the scattering formalism developed in the course, to elementary scattering theory of a non-relativistic particle on a potential in single-particle quantum mechanics. The Hamilton operator is given by

$$H = \frac{\vec{P}^{\,2}}{2m} + V(\vec{X}) \tag{3}$$

and the potential is assumed to have finite range, that is, $|\vec{x}| V(\vec{x}) \to 0$ for $|\vec{x}| \to \infty$.

a) As a first step, define

$$G_\pm(\vec{x}, \vec{x}\,') \equiv \langle \vec{x}| \frac{1}{E_{\vec{p}} - \widetilde{H}_0 \pm i\varepsilon} |\vec{x}\,'\rangle \tag{4}$$

and prove that

$$G_\pm(\vec{x}, \vec{x}\,') = -\frac{m}{2\pi} \frac{e^{\pm ip|\vec{x}-\vec{x}\,'|}}{|\vec{x} - \vec{x}\,'|} \tag{5}$$

where $E_{\vec{p}} = p^2/(2m)$.

---

[1] The * means to be handed in.

b)* Demonstrate, starting from the Lippmann-Schwinger equation in the position-space representation of the states and using the result of part **a)**, that the in- and out-states for the scattering of a particle with momentum $\vec{p}$ have the asymptotic behavior

$$\langle\vec{x}|\psi_{\vec{p}}^{\text{in/out}}\rangle = e^{i\vec{p}\cdot\vec{x}} + f_{\pm}(\vec{p}\,',\vec{p})\frac{e^{\pm ipr}}{r} + O\left(\frac{1}{r^2}\right) \tag{6}$$

where the scattering amplitudes are expressed in terms of the potential and the in- and out-wave functions as

$$f_{\pm}(\vec{p}\,',\vec{p}) = -\frac{m}{2\pi}\int d^3x' e^{\mp i\vec{p}\,'\cdot\vec{x}\,'}V(\vec{x}\,')\langle\vec{x}\,'|\psi_{\vec{p}}^{\text{in/out}}\rangle = -\frac{m}{2\pi}\langle\pm\vec{p}\,'|V|\psi_{\vec{p}}^{\text{in/out}}\rangle\,. \tag{7}$$

Here $r = |\vec{x}|$ and $\vec{p}\,'$ is defined as a vector in the direction of $\vec{x}$ with modulus equal to $|\vec{p}|$.

c)* Using the results of Problem 1 as well as of part **b)**, establish the relation

$$S_{\vec{p}_b\vec{p}_a} = \langle\psi_{\vec{p}_b}^{\text{out}}|\psi_{\vec{p}_a}^{\text{in}}\rangle = \delta^{(3)}(\vec{p}_b - \vec{p}_a) + 2\pi i\,\delta(E_b - E_a)\frac{2\pi}{m}f_+(\vec{p}_b,\vec{p}_a) \tag{8}$$

between the S-matrix elements and the scattering amplitude. Here $E_i \equiv \vec{p}_i^{\,2}/(2m)$.
Hint: you will find the identity

$$\lim_{t\to\infty}\frac{e^{i(E_b-E_a)t}}{E_b - E_a - i\epsilon} = 2\pi i\,\delta(E_b - E_a) \tag{9}$$

useful. Prove it.

d)* Then prove the identities

$$\frac{d\sigma}{d\Omega} = |f_+(\vec{p}_b,\vec{p}_a)|^2, \qquad \sigma_{\text{tot}} = \frac{4\pi}{|\vec{p}_a|}\,\text{Im}\,f_+(\vec{p}_a,\vec{p}_a). \tag{10}$$

The second relation for the total cross section is also known as the "optical theorem."
Hint: To obtain the first of Eqs. (10), recall the quantum mechanics result

$$\frac{d\sigma_{ba}}{d\Omega} = \left(\frac{m}{2\pi}\right)^2|T_{ba}|^2 \tag{11}$$

where T_{ba} is the transition rate. To derive the second of Eqs. (10), prove and employ the identity

$$\text{Im}\,\frac{1}{x \pm i\epsilon} = \mp\,\pi\delta(x). \tag{12}$$

3 Two-body decay

Consider a spinless particle of mass M which decays into two particles of masses m_1 and m_2, respectively, also with spin 0. Assume that the T-matrix element $T_{\beta\alpha}$ takes the same constant value λ irrespective of the momenta of the particles. Compute the total decay width in the rest frame of the decaying particle and express the result in terms of the masses. For such a decay to be kinematically possible, $M > m_1 + m_2$ should hold. How does this condition arise in the calculation?

4 Three-body decay [discussed at central tutorial of 05.12]

Consider the process of *muon decay*, $\mu^- \to e^- \bar{\nu}_e \nu_\mu$. The amplitude squared, summed over the spin states of the decay products and averaged over the spin states of the initial muon, is

$$|\mathcal{A}_{fi}|^2 = 64\, G_F^2 (k_1 \cdot k_2')(k_1' \cdot k_3'), \tag{13}$$

where G_F is the *Fermi constant*, k_1 is the four-momentum of the muon, and $k'_{1,2,3}$ are the four-momenta of the $\bar{\nu}_e, \nu_\mu$ and e^-, respectively. In the rest frame of the muon, its decay rate is therefore

$$\Gamma = \frac{32\, G_F^2}{m} \int (k_1 \cdot k_2')(k_1' \cdot k_3') d\Pi_3(k_1), \tag{14}$$

where

$$d\Pi_n(k) \equiv (2\pi)^4 \delta^{(4)}\Big(k - \sum_{j=1}^{n} k_j'\Big) \prod_{j=1}^{n} \frac{d^3 k_j'}{(2\pi)^3} \frac{1}{2E_{k_j'}}\,, \tag{15}$$

while $k_1^\mu = (m, \mathbf{0})$ with m the muon mass. All the final state particles can be taken massless.
In this problem we will evaluate Γ through the following analysis:

a) Show that

$$\Gamma = \frac{32\, G_F^2}{m} \int \frac{d^3 k_3'}{(2\pi)^3} \frac{1}{2E_{k_3'}} k_{1\mu} k'_{3\nu} \int k_2'^{\mu} k_1'^{\nu} d\Pi_2(k_1 - k_3'). \tag{16}$$

b) Use Lorentz invariance to argue that

$$\int k_2'^{\mu} k_1'^{\nu} d\Pi_2(q) = A q^2 g^{\mu\nu} + B q^\mu q^\nu\,, \tag{17}$$

where A and B are numerical constants.

c) Show that

$$\int d\Pi_2(q) = \frac{1}{8\pi}\,. \tag{18}$$

Then, by contracting both sides of Eq. (17) with $g_{\mu\nu}$ and $q_\mu q_\nu$ and using Eq. (18), evaluate A and B.

d) Plug the results obtained in **b)** and **c)** into Eq. (16) and compute $d\Gamma/dE_e$, where $E_e \equiv E_{k_3'}$ is the electron energy. Note that the maximum value of E_e is reached when the electron is emitted in one direction, and the two neutrinos in the opposite direction. What is this maximum value?

e) Perform the integral over E_e to obtain the muon decay rate Γ. Using the measured values of the lifetime of the muon, 2.197 μs, and of the muon mass, 105.66 MeV, determine the value of G_F in GeV^{-2}.

f) Define the *energy spectrum* of the electron as $P(E_e) \equiv \Gamma^{-1} d\Gamma/dE_e$. Note that $P(E_e)dE_e$ is the probability for the electron to be emitted with energy between E_e and $E_e + dE_e$. Draw a graph of $P(E_e)$ versus E_e/m.

5 [discussed at central tutorial of 05.12]

Let $(\mathcal{H}, U, \Omega, \phi, D)$ be a Wightman theory as in the lecture such that $\langle \Omega | \phi(x) \Omega \rangle = 0$. Suppose that the Källén-Lehmann representation has the form

$$\langle \Omega | \phi(x)\phi(y)\Omega \rangle = \int d\rho(M^2)\Delta_+(x-y; M^2), \tag{19}$$

$$d\rho(M^2) = Z\delta(M^2 - m^2)d(M^2) + d\tilde{\rho}(M^2), \tag{20}$$

where $Z > 0$ and $m > 0$ are fixed and $d\tilde{\rho}$ is supported in $[\tilde{m}^2, \infty)$ for some $\tilde{m} > m$. Let $h \in S$ be such that $\widehat{h}$ is supported near the mass hyperboloid H_m. Show that

$$(\Box_x + m^2)\phi(h_x)\Omega = 0. \tag{21}$$

Hints:

1) The solution does not have to be completely rigorous. You can exchange the order of various mathematical operations without justification.

2) For the definition of the Schwartz class of test-functions S, see the notes of Math Lecture 1.

Some relevant definitions:

1. In this problem we use notation which is common in the mathematical literature, namely a vector in Hilbert space is denoted Ψ and not $|\Psi\rangle$. For a scalar product of two vectors Ψ_1, Ψ_2 we write $\langle \Psi_1 | \Psi_2 \rangle$ which coincides with the physics notation. However, given a linear operator A, the scalar product of Ψ_1 and $A\Psi_2$ has the form $\langle \Psi_1 | A\Psi_2 \rangle$ and not $\langle \Psi_1 | A | \Psi_2 \rangle$.
2. $\Delta_+(x-y; M^2) = \int \frac{d^3p}{(2\pi)^3 2p^0} e^{-ip(x-y)}$ with $p^0 = \sqrt{\vec{p}^{\,2} + M^2}$.
3. $\widehat{h}(q) = \frac{1}{2\pi} \int d^4q e^{iqx} h(x)$ with $qx = q^0x^0 - \vec{q} \cdot \vec{x}$.
4. $H_m = \{p \in \mathbb{R}^4 | p^2 = m^2, p^0 \geq 0\}$.
5. "Near the mass hyperboloid H_m" means here in the set

$$M_\varepsilon = \{q \in \mathbb{R}^4 |\, |q^2 - m^2| < \varepsilon, q^0 \geq 0\}, \tag{22}$$

   for some $\varepsilon > 0$ much smaller than $(\tilde{m}^2 - m^2)$.

6. $h_x(y) = h(x-y)$.