Quantum Field Theory WS 2018/2019

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http://www.t75.ph.tum.de/teaching/ws18-quantum-field-theory/

Sheet 5: Feynman rules
(21.11.2018; solution due by 28.11 at 16:00, the parts required for hand-in will be announced on 27.11 at 8:00am; discussed at tutorials of 28.11, 29.11 and 03.12)

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1 Feynman rules and ϕ^4

a) For the theory of a real scalar field with interaction Lagrangian $\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4$ derive the general form of the Schwinger-Dyson equations i.e. determine the right-hand side of

$$(\partial^2_{(x)} + m^2)\langle\Omega|T(\phi(x)\phi(x_1)\ldots\phi(x_n))|\Omega\rangle \;=\; ?\,. \tag{1}$$

b)* [1] Show that the equation-of-motion identity

$$(\partial^2_{(x)} + m^2)\langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle \;=\; -\frac{\lambda}{3!}\langle\Omega|T(\phi(x)^3\phi(y))|\Omega\rangle - i\delta^{(4)}(x-y) \tag{2}$$

holds here at order $O(\lambda)$ and $O(\lambda^2)$ by explicitly computing the left- and right-hand side with the help of the Feynman rules.

2* Effective action

We define the functional

$$\Gamma[\varphi] \;=\; W[J] - \int \mathrm{d}^4x\,\varphi(x)J(x) \tag{3}$$

as the Legendre transform of the generating functional of connected Green functions, that is $\varphi(x)$ is related to $J(x)$ through

$$\varphi(x) \;=\; \frac{\delta W[J]}{\delta J(x)}\,. \tag{4}$$

a) Compute $\Gamma[\varphi]$ explicitly for the field theory of the free, real scalar field and show that is coincides with the action $S[\varphi]$.

b) Now show that $\Gamma[\varphi] = S[\varphi]$ holds also in the interacting theory in the classical limit $\hbar \to 0$.

[Hint: Reinstating $\hbar$, the generating functional reads

$$Z[J] \;=\; e^{\frac{i}{\hbar}W[J]} \;=\; |N|^2 \int \mathcal{D}[\varphi] \exp\left[\frac{i}{\hbar}\left(S[\varphi] + \int \mathrm{d}^4x\,\varphi(x)J(x)\right)\right]\,. \tag{5}$$

Now apply the method of a stationary phase.]

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[1] The * means to be handed in.

3 Math problem [discussed at central tutorial of 05.12]

Show that for $f_n \to 0$ in the Schwartz space of test-functions S we have

$$\int \mathrm{d}^4p\, \bar{\hat{f}}_n(p) \frac{1}{|p^2| + m^2} \hat{f}_n(p) \;\to\; 0\;, \tag{6}$$

where $|p^2| := p_0^2 + p_1^2 + p_2^2 + p_3^2$ denotes the Euclidean norm-squared.
Hints:

1) $\hat{f} \in S$ is the Fourier transform of $f \in S$ i.e.

$$\hat{f} \;=\; \frac{1}{(2\pi)^2} \int \mathrm{d}^4x\, e^{-ip\cdot x} f(x) \tag{7}$$

with the Euclidean scalar product $p \cdot x = p_0x_0 + p_1x_1 + p_2x_2 + p_3x_3$. You can use that the Fourier transform satisfies $\langle \hat{f}_1 | \hat{f}_2 \rangle = \langle f_1 | f_2 \rangle$ for all $f_1, f_2 \in S$, where $\langle f_1 | f_2 \rangle = \int \mathrm{d}^4x \bar{f}_1(x) f_2(x)$ (Plancherel theorem).

2) Recall the $f_n \to 0$ in S means that $||f_n||_{\alpha\beta} \to 0$ for all multiindices α, β. Here $||f_n||_{\alpha\beta} := \sup_{x\in\mathbb{R}^4} |x^\alpha \partial^\beta f(x)|$ (see the notes from Math Lecture 1).