Quantum Field Theory WS 2018/2019

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http://www.t75.ph.tum.de/teaching/ws18-quantum-field-theory/

Sheet 3: the path integral
(07.11.2018; solution due by 14.11 at 16:00, the parts required for hand-in will be announced on 13.11 at 8:00am; discussed at tutorials of 14.11, 15.11 and 19.11)

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1. q- and p-eigenstates and Gauss integral

a) Show that it is possible to choose the q- and p-eigenstates such that

$$p\langle q;t|p;t\rangle = \frac{1}{i}\frac{\partial}{\partial q}\langle q;t|p;t\rangle \tag{1}$$

holds for a pair of canonically conjugated coordinates using only the commutation relations and the fact that $|q\rangle$ and $|p\rangle$ are coordinate and momentum eigenstates, respectively.

b)* [1] Prove the formula

$$\begin{aligned}\int_{-\infty}^{\infty}\prod_k \mathrm{d}z_k \exp\left(-Q(z)\right) &= \left(\det\frac{A}{2\pi}\right)^{-1/2}\exp\left(\frac{1}{2}[A^{-1}]_{kl}B_kB_l - C\right)\\ &= \left(\det\frac{A}{2\pi}\right)^{-1/2}\exp\left(-Q(\bar{z})\right)\end{aligned} \tag{2}$$

for the N-dimensional Gauss integral. Here $Q(z) = \frac{1}{2}A_{kl}z_kz_l + B_kz_k + C$ (with $k,l = 1,\dots,N$) is a quadratic form and $\bar{z}$ denotes its stationary point.

2. Path integral

a)* Show that for complex scalar fields

$$\begin{aligned}&\int \mathcal{D}\phi^*\mathcal{D}\phi\,\exp\left[i\int \mathrm{d}^4x\,\mathrm{d}^4y\,[\phi^*(x)M(x,y)\phi(y)] + i\int \mathrm{d}^4x\,[J^*(x)\phi(x)+\phi^*(x)J(x)]\right]\\ &= \frac{\mathcal{N}}{\det M}\exp\left(-i\int \mathrm{d}^4x\,\mathrm{d}^4y\,J^*(x)M^{-1}(x,y)J(y)\right)\end{aligned} \tag{3}$$

for some (infinite) constant $\mathcal{N}$.

b) In this problem, you will explicitly construct all the states that satisfy

$$\hat{\phi}(\vec{x})|\Phi\rangle = \Phi(\vec{x})|\Phi\rangle\,. \tag{4}$$

This is one way to define the measure on the path integral.

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[1] The * means to be handed in.

(a) * Write the eigenstates of $\hat{x} = c(a + a^\dagger)$ for a single harmonic oscillator in terms of creation operators acting on the vacuum. That is, find $f_z(a^\dagger)$ such that

$$\hat{x}|\psi\rangle \;=\; z|\psi\rangle \quad \text{where} \quad |\psi\rangle = f_z(a^\dagger)|0\rangle \; . \tag{5}$$

(b) * Generalize the above construction to field theory, to find the eigenstates $|\Phi\rangle$ of $\hat{\phi}(\vec{x})$ that satisfy $\hat{\phi}(\vec{x})|\Phi\rangle = \Phi(\vec{x})|\Phi\rangle$.

(c) Prove that these eigenstates satisfy the orthogonality relation

$$\langle\Phi'|\Phi\rangle \;=\; \int \mathcal{D}\Pi \, \langle\Phi'|\Pi\rangle \, \langle\Pi|\Phi\rangle \;=\; \int \mathcal{D}\Pi \, \exp\left(-i\int \mathrm{d}^3x \, \Pi(\vec{x}) \, [\Phi(\vec{x}) - \Phi'(\vec{x})]\right) \; , \tag{6}$$

where $|\Pi\rangle$ are the conjugate states of $|\Phi\rangle$, i.e.

$$\langle\Pi|\Phi\rangle = \exp\left(-i\int \mathrm{d}^3x \, \Pi(\vec{x})\Phi(\vec{x})\right) \quad \text{and} \quad \hat{\pi}(\vec{x})|\Pi\rangle \;=\; \Pi(\vec{x})|\Pi\rangle \; . \tag{7}$$

As a remark, eq. (6) is the generalization of

$$\langle\vec{x}\,'|\vec{x}\,\rangle \;=\; \delta(\vec{x} - \vec{x}\,') \;=\; \frac{1}{2\pi}\int \mathrm{d}p \, \exp\left(-i\vec{p}\cdot(\vec{x} - \vec{x}\,')\right) \; , \tag{8}$$

and eq. (7) is the generalization of $\langle\vec{p}\,|\vec{x}\,\rangle = \exp\left(-i\vec{p}\cdot\vec{x}\right)$.

3. Quantum statistical mechanics [discussed at central tutorial of 14.11]

a) Evaluate the quantum statistical partition function $Z = \mathrm{tr}\left[\exp\left(-\beta H\right)\right] = \int \mathrm{d}x \langle x| \exp\left(-\beta H\right) |x\rangle$ where $\beta = 1/kT$ is the inverse temperature, using the path integral, i.e. in analogy to evaluating the matrix elements of $\exp\left(-iHt\right)$ in terms of functional integrals. Show that one again finds a functional integral, over functions defined on a domain that is of length β and periodically connected in the time direction. Note that the Euclidean form of the Lagrangian L_{E} appears in the weight, i.e. the Hamiltonian H and the Lagrangian L_{E} have the following forms

$$H = \frac{p^2}{2m} + V(x) \quad \text{and} \quad L_{\mathrm{E}} = \frac{m}{2}\dot{x}^2 + V(x(t)) \,. \tag{9}$$

b) Evaluate this integral for a simple harmonic oscillator $L_{\mathrm{E}} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2$, by introducing a Fourier decomposition of $x(t)$, i.e.

$$x(t) = \sum_n x_n \frac{1}{\sqrt{\beta}} \exp\left(2\pi i n t/\beta\right) \,, \tag{10}$$

where $x_{-n} = x_n^*$ to make $x(t)$ real. To do so, you have to verify that the Euclidean action reads

$$S_{\mathrm{E}} = \int_0^\beta \mathrm{d}t\, L_{\mathrm{E}} = \frac{1}{2}\omega^2 x_0^2 + \sum_{n=1}^{\infty} \left(\frac{(2\pi n)^2}{\beta^2} + \omega^2\right) |x_n|^2 \,, \tag{11}$$

such that the path integral will be Gaussian. The final result should read

$$Z \propto \frac{\exp\left(-\beta\omega/2\right)}{1 - \exp\left(-\beta\omega\right)} \,. \tag{12}$$

The dependence of the result on β is a bit subtle to obtain explicitly, since the measure for the integral over $x(t)$ depends on β in any discretization. However, the dependence on ω should be unambiguous. Show that, up to a (possibly divergent and β-dependent) constant, the integral reproduces exactly the familiar expression for the quantum partition function of an oscillator.

Hint: you might need the identity

$$\frac{1}{2}\left(\exp(z) - \exp(-z)\right) = \mathrm{sinh} z = z \prod_{n=1}^{\infty}\left(1 + \frac{z^2}{(n\pi)^2}\right) \,. \tag{13}$$

c) Generalize this construction to field theory. Show that the quantum statistical partition function for a free scalar field can be written in terms of a functional integral. The value of this integral is given by

$$\left[\det\left(-\partial^2 + m^2\right)\right]^{-\frac{1}{2}} \,, \tag{14}$$

where the operator acts on functions on Euclidean space that are periodic in the time direction with periodicity β. As before, the β dependence of this expression is difficult to compute directly. However, the dependence on m^2 is unambiguous. (More generally, one can usually evaluate the variation of a functional determinant with respect to any explicit parameter in the Lagrangian.) Show that the determinant indeed reproduces the partition function for relativistic scalar particles.