Quantum Field Theory WS 2018/2019

Prof. Andreas Weiler (TUM); Dr. Patrick Vaudrevange,
Dr. Ennio Salvioni, Dr. Javi Serra
http://www.t75.ph.tum.de/teaching/ws18-quantum-field-theory/

Sheet 2: more on canonical quantization
(31.10.2018; solution due by 07.11 at 16:00, the parts required for hand-in will be announced on 06.11 at 8:00am; discussed at tutorials of 07.11, 08.11 and 12.11)

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1 Non-relativistic fields

Consider a *non-hermitian* scalar field $\Phi(x)$ with Lagrangian density $\mathcal{L} = -\Phi^\dagger(\Box + m^2)\Phi$.

a)*[1] Decomposing the field as $\Phi(x) = e^{-imt}\phi(x)/\sqrt{2m}$, show that in the non-relativistic limit the Lagrangian can be written as

$$\mathcal{L} = \phi^\dagger\Big(i\partial_0 + \frac{\nabla^2}{2m}\Big)\phi\,. \tag{1}$$

Now make an ansatz of purely destruction field,

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} e^{-ip\cdot x} a(\vec{p}). \tag{2}$$

Prove that in this case it must be $p^0 = \vec{p}^{\,2}/(2m)$.

b)* Use the results from the lecture (discussion of the non-hermitian scalar field) to show that relativistic causality is violated, that is, the field commutator does not vanish for space-like distances.

c)* Consider the wavefunction $\psi(x) = \langle \vec{x}, t|\Psi_1\rangle$, where $|\Psi_1\rangle$ is an arbitrary single-particle state in the Fock space,

$$|\Psi_1\rangle = \int \frac{d^3p}{(2\pi)^3}\,\psi(\vec{p})|p\rangle\,, \qquad |p\rangle = a(\vec{p})^\dagger|0\rangle\,, \tag{3}$$

with $\psi(\vec{p})$ the momentum-space wavefunction, and $|\vec{x}, t\rangle = \phi(x)^\dagger|0\rangle$ denotes a position eigenstate. Show that $\psi(x)$ satisfies the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(t,\vec{x}) = H_0\psi(t,\vec{x}), \qquad H_0 = -\nabla^2/(2m). \tag{4}$$

What would be the corresponding equation for a two-particle state with wavefunction $\psi(\vec{x},\vec{y},t) = \langle\vec{x},\vec{y};t|\Psi_2\rangle$, where $|\Psi_2\rangle$ is an arbitrary two-particle state defined analogously to $|\Psi_1\rangle$? Sketch the derivation. Hence, what kind of system does this field theory describe?

d) Looking at the Lagrangian (1), are you able to find an object that you can identify with the (conserved) total probability of non-relativistic quantum mechanics? Use this to argue that a hermitian scalar field does not have a viable interpretation in this limit.

e) If the Lagrangian (1) describes a fermion field ψ instead of a scalar ϕ, the fermion field must satisfy the canonical *anticommutation* relations

$$\{\psi(t,\vec{x}), \pi(t,\vec{y})\} = i\,\delta^{(3)}(\vec{x}-\vec{y})\,, \tag{5}$$
$$\{\psi(t,\vec{x}), \psi(t,\vec{y})\} = \{\pi(t,\vec{x}), \pi(t,\vec{y})\} = 0\,. \tag{6}$$

Repeat the analysis of points **a)** to **d)**, and find out which result changes for a fermionic field.

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[1] The * means to be handed in.

2 Correlation functions in free (hermitian) scalar theory

Derive expressions of the form $\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x_1-x_2)}\Delta(p)$ for the two-point functions

a) $\langle 0|\phi(x_1)\phi(x_2)|0\rangle$,

b) $\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle$,

c)* $\langle 0|T\{\partial_\mu\phi(x_1)\partial_\nu\phi(x_2)\}|0\rangle$,

of the *free*, real scalar field, by expressing the fields in terms of annihilation and creation operators and evaluating the matrix elements. Here $|0\rangle$ is the Fock space vacuum and T denotes time ordering.

3 Wightman theory [discussed at central tutorial of 14.11]

Consider a Wightman theory $(\mathcal{H}, U, \Omega, \phi, D)$ as in the lecture. Show rigorously, using the hints below, that for any $\Psi_1, \Psi_2 \in D \cap D(P_\mu)$ we have

$$\langle\Psi_1|i[P_\mu, \phi(f)]\Psi_2\rangle = \langle\Psi_1|(\partial_\mu\phi)(f)\Psi_2\rangle\,, \quad f \in S\,. \tag{7}$$

Check this equality in the free scalar field by an explicit (possibly non-rigorous) computation.
Hints:

1. $D(P_\mu) := \{\Psi \in \mathcal{H}\,|\,\lim_{s\to 0}\frac{e^{isP_\mu}-1}{s}\,\Psi \text{ exists}\}$.
2. Derivatives of distributions are defined according to $(\partial_\mu\phi)(f) := \phi(-\partial_\mu f)$ (consistently with integration by parts).
3. For any $\varphi \in S'$ and $f \in S$ the function $\mathbb{R}^4 \ni a \mapsto \varphi(f_a)$, where $f_a(x) = f(x-a)$, $a = \{a^\mu\}_{\mu=0,1,2,3}$, is partially differentiable at zero and

$$\frac{\partial}{\partial a^\mu}\,\varphi(f_a)|_{a=0} = -\varphi(\partial_\mu f). \tag{8}$$