Quantum Field Theory WS 2018/2019

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http://www.t75.ph.tum.de/teaching/ws18-quantum-field-theory/

Bonus-Sheet 14
Discussed at central tutorial of **20.02** at 14:15

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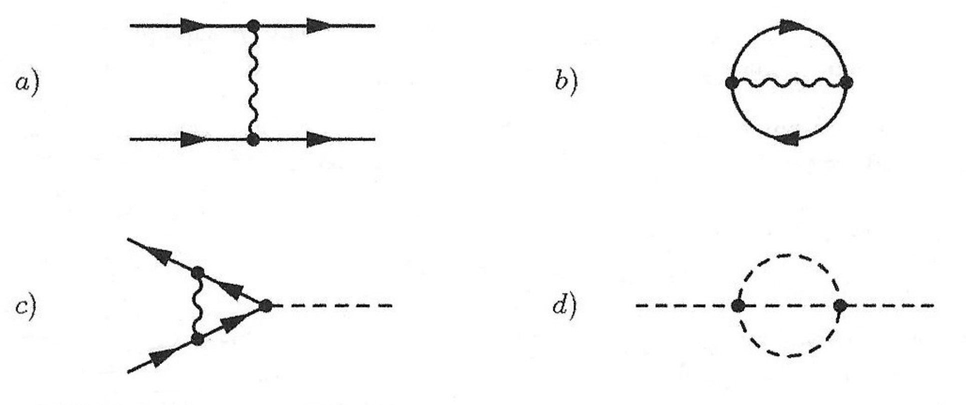
1 Some Feynman graphs

In the following diagrams a solid (dashed, wavy) line represents the propagator of a Dirac field (neutral scalar field, massive vector field). The vertex factor for a Dirac field coupling to the vector field is $i\,e\,\gamma^\mu_{\alpha\beta}$; for the coupling to the scalar field $i\,g\,\delta_{\alpha\beta}$; the interaction Langrangian of the scalar self-coupling is $-i\,\lambda/4!\,\phi^4$. For each of the four diagrams shown below write down its expression according to the momentum space Feynman rules for scattering amplitudes (that is, amputated Green functions), eliminating as many delta-functions as possible.



2 Loop correction

a) Consider the following expression for the fermion-loop correction to the two-point function of $A_\mu(x)$:

$$i\Pi^{\mu\nu}(p) \;\equiv\; (-1)\,(i\,e)^2\,i^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{tr}[\gamma^\mu(\not{k}+m)\gamma^\nu(\not{p}+\not{k}+m)]}{(k^2-m^2+i\,\epsilon)((p+k)^2-m^2+i\,\epsilon)}\,, \tag{1}$$

where p denotes the external momentum. Show that $p_\mu \Pi^{\mu\nu}(p) = p_\nu \Pi^{\mu\nu}(p) = 0$ without calculating the integral explicitly and without even introducing a Feynman parameter.
b) Derive the result for the one-loop photon vacuum polarization in the on-shell scheme,

$$\Pi(q^2) \;=\; \frac{2\alpha_{\mathrm{em}}}{\pi} \int_0^1 \mathrm{d}x\, x\bar{x} \ln \frac{m^2 - x\bar{x}q^2 - i\,\epsilon}{m^2}\,, \tag{2}$$

where $\bar{x}$ is defined here as $\bar{x} = 1 - x$ and then show that the leading term at small q^2 is given by

$$\Pi(q^2) \;=\; -\frac{\alpha_{\mathrm{em}}}{15\pi}\frac{q^2}{m^2}\,. \tag{3}$$

Note: If your computation leads to a different Feynman parameter integral for $\Pi(q^2)$, show analytically that it is equivalent.

c) Let the electrostatic potential between the electron and the proton in the hydrogen atom be given by

$$V(\vec{r}) \;=\; \int \frac{\mathrm{d}^3\vec{q}}{(2\pi)^3}\, e^{i\,\vec{q}\cdot\vec{r}} \frac{-e^2}{\vec{q}^2(1-\Pi(-\vec{q}^2))} \,. \tag{4}$$

Show that for $r \gg 1/m$ the photon vacuum polarization gives a correction $\delta V(\vec{r}) \propto \alpha_{\mathrm{em}}^2 \delta^{(3)}(\vec{r})$ to the Coulomb potential and calculate the proportionality constant. Then, evaluate the correction to the energy levels of hydrogen from $\delta V(\vec{r})$ and provide an estimate for the contribution to the Lamb shift of the $2s2p$ levels in MHz. Can this account for the observed Lamb shift? What other contributions could there be?

d) Running of the coupling constant α_{em} in QED with one charged Dirac fermion (electron): Using the relation $Z_e = 1/\sqrt{Z_3}$ to be shown in Ex.3 and using the $\overline{\mathrm{MS}}$ scheme, show that the one-loop beta function is given by

$$\beta(\alpha_{\mathrm{em}}) \;=\; \frac{2\,\alpha_{\mathrm{em}}^2}{3\pi} + \mathcal{O}(\alpha_{\mathrm{em}}^3) \,. \tag{5}$$

Discuss the behaviour of QED in the IR and UV.

3 QED Ward identity

Prove the QED Ward identity

$$F_1^0(q^2=0,m) \;=\; \frac{1}{Z_2} \,. \tag{6}$$

Using $1/Z_e = \sqrt{Z_3}\, Z_2\, F_1^0(q^2=0,m)$, eq. (6) implies $Z_e = 1/\sqrt{Z_3}$, i.e. a relation between charge and photon field renormalization.

We use the definitions

$$\Psi_0(x) \;=\; \sqrt{Z_2}\,\Psi(x) \quad , \quad A_0^\mu(x) \;=\; \sqrt{Z_3}\,A^\mu(x) \quad , \quad e_0 \;=\; Z_e\, e \,. \tag{7}$$

$F_1^0(q^2=0,m)$ is computed from the amputated electron-photon three-point function

$$\begin{aligned}\langle e^-(p',s')\,\gamma(q,\lambda);\mathrm{out}\,|\,e^-(p,s);\mathrm{in}\rangle \;&=\; (2\pi)^4\delta^{(4)}(p'+q-p)\,\sqrt{Z_3}\,Z_2\,i\,e_0 \\ &\quad\; \bar{u}(p',s')\,\Gamma_0^\mu(p',p)\,u(p,s)\,\epsilon_\mu^*(q,\lambda) \,. \end{aligned} \tag{8}$$

Then, $\Gamma_0^\mu(p',p)$ must be a 4×4 matrix in Dirac spinor space and one can show that it must be of the form

$$\Gamma_0^\mu(p',p) \;=\; \gamma^\mu\, F_1^0(q^2,m) + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}\, F_2^0(q^2,m) \,. \tag{9}$$

Further hints: Use the Ward identity for the Noether current of a gauge symmetry ("QFT Noether theorem" on page 105 of the lecture notes) inserted into the electron two-point function, where $j_0^\mu = -\bar{\Psi}_0\gamma^\mu\Psi_0$, $F_\Psi = i\,\Psi$ and $F_{\bar{\Psi}} = -i\,\bar{\Psi}$.