Quantum Field Theory WS 2018/2019

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http://www.t75.ph.tum.de/teaching/ws18-quantum-field-theory/

Sheet 12: fermions
(23.01.2019; solution due by 30.01.2019 at 16:00, the parts required for hand-in will be announced on 29.01.2019 at 8:00am; discussed at tutorials of 30.01, 31.01 and 04.02)

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1 Properties of gamma matrices

Prove the following identities, using only the definining property $\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}$ and the definitions $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = (-i/4!)\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ and $\sigma^{\mu\nu} \equiv (i/2)[\gamma^\mu, \gamma^\nu]$, i.e. without resorting to a particular representation:

a)

1. $\gamma^\mu\gamma_\mu = 4$
2. $\gamma_\mu\, \gamma^\nu\, \gamma^\mu = -2\gamma^\nu$ and $\gamma_\mu\, \gamma^\nu\gamma^\rho\gamma^\mu = 4\, g^{\nu\rho}\,,$
3. $\gamma_\mu\, \gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu = -2\, \gamma^\sigma\gamma^\rho\gamma^\nu\,,$
4. $\gamma_\mu\sigma^{\nu\rho}\gamma^\mu = 0\,,$
5. $\mathrm{Tr}\, \gamma^\mu = 0\,,$
6. $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\, g^{\mu\nu},$
7. * [1] $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}),$
8. * $\mathrm{Tr}(\gamma^\mu \ldots \gamma^\sigma) = 0$ for an odd number of γ matrices;

b)

1. $(\gamma_5)^2 = 1$ and $\mathrm{Tr}\, \gamma_5 = 0\,,$
2. $\{\gamma^\mu, \gamma_5\} = 0$ and $[\sigma^{\mu\nu}, \gamma_5] = 0\,,$
3. $\mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu) = 0\,,$
4. $\mathrm{Tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\, \epsilon^{\mu\nu\rho\sigma}\,,$
5. * $\gamma_5\sigma^{\mu\nu} = \frac{i}{2}\, \epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma}\,,$
6. * $\gamma^\mu\gamma^\nu\gamma^\rho = g^{\nu\rho}\gamma^\mu - g^{\mu\rho}\gamma^\nu + g^{\mu\nu}\gamma^\rho + i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5\,.$

Hint: Some useful tricks include exploiting the cyclicity of the trace, inserting $(\gamma_5)^2 = 1$ into a trace, and using the identity $\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\mu\nu\rho\sigma} = -\delta^\mu_{[\alpha}\delta^\nu_\beta\,\delta^\rho_\gamma\,\delta^\sigma_{\delta]}\,.$

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[1] The * means to be handed in.

2* Weyl spinors

Given the generators of rotations $J^i = (1/2)\epsilon^{ijk}J^{jk}$ and boosts $K^i = J^{i0}$, where $i, j, k = 1, 2, 3$, infinitesimal Lorentz transformations can be written as

$$\psi \to (\mathbf{1} - i\boldsymbol{\theta} \cdot \boldsymbol{J} + i\boldsymbol{\eta} \cdot \boldsymbol{K})\psi\,. \tag{1}$$

Recall from the lecture the definition of the complexified Lorentz generators $\boldsymbol{A} = (\boldsymbol{J} + i\boldsymbol{K})/2$ and $\boldsymbol{B} = (\boldsymbol{J} - i\boldsymbol{K})/2$, which separately fulfill the commutation relations of angular momentum and commute with each other. Any finite irreducible representation generated by $\boldsymbol{A}$ is locally isomorphic to a representation generated by a usual angular momentum, i.e. locally isomorphic to a representation of $SU(2)$; similarly for $\boldsymbol{B}$. Therefore all finite-dimensional representations of the Lorentz group correspond to pairs (b, a) of integers or half-integers.
Note: Since the $\boldsymbol{A}, \boldsymbol{B}$ are non-hermitian, $\boldsymbol{A}^\dagger = \boldsymbol{B}$, the *global* structure of the representations is however a non-unitary analytic continuation of the corresponding $SU(2)$ representations. $\boldsymbol{A}, \boldsymbol{B}$ generate the group $\mathrm{Spin}(1, 3) \cong SL(2, \mathbb{C})$, which is in turn the (universal) double cover of the proper orthochronous Lorentz group $SO(1, 3)$.

a) Now consider the simplest non-trivial representations. Those are the left- and right-handed Weyl spinors $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. Use the fact that spin-1/2 representations of angular momentum are generated by $\boldsymbol{\sigma}/2$ to show that the Weyl spinors transform as

$$\begin{aligned}
\psi_L \to D_L(\Lambda)\psi_L &= \left(\mathbf{1} - (i\boldsymbol{\theta} + \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right)\psi_L\,, \\
\psi_R \to D_R(\Lambda)\psi_R &= \left(\mathbf{1} - (i\boldsymbol{\theta} - \boldsymbol{\eta}) \cdot \frac{\boldsymbol{\sigma}}{2}\right)\psi_R\,.
\end{aligned} \tag{2}$$

Use $\boldsymbol{\sigma}^* = -\sigma^2\boldsymbol{\sigma}\sigma^2$ and the explicit form of $D_{L,R}(\Lambda)$ to show that $\sigma^2 D_L(\Lambda)^*\sigma^2 = D_R(\Lambda)$. Show how one can infer from this that if $\psi_L \in (\frac{1}{2}, 0)$, then $\sigma^2\psi_L^*$ is a right-handed Weyl spinor, i.e. $\sigma^2\psi_L^* \in (0, \frac{1}{2})$.

b) Based on the results of the previous point, show that two distinct types of fermion masses can be written compatibly with Lorentz invariance,

$$\mathcal{L}_D = m_D\psi_L^\dagger\psi_R + \text{h.c.}, \qquad \mathcal{L}_M = m_M\psi_L^T i\sigma^2\psi_L + \text{h.c.}, \tag{3}$$

where for the second (Majorana) type, an analogous term could be written for ψ_R. What does each of the two terms in Eq. (3) imply, as far as internal symmetries are concerned? Finally, what happens to $\mathcal{L}_M$ if the components of ψ_L commute with each other? As you will see later in the lecture, this apparent puzzle is resolved by treating the spinors as anticommuting (Grassmann) variables.

c) Prove that if ψ_R and ξ_R are right-handed Weyl spinors and $\sigma^\mu \equiv (1, \boldsymbol{\sigma})$, then $U^\mu = \xi_R^\dagger\sigma^\mu\psi_R$ is a Lorentz four-vector. Show the same for $V^\mu = \xi_L^\dagger\bar{\sigma}^\mu\psi_L$, where ψ_L and ξ_L are left-handed Weyl spinors and $\bar{\sigma}^\mu \equiv (1, -\boldsymbol{\sigma})$.

d) Verify explicitly that for $D_L(\Lambda) = \exp(-i\theta\boldsymbol{n} \cdot \boldsymbol{\sigma}/2)$, $L[D_L(\Lambda)]$ is a rotation by the angle θ around $\boldsymbol{n}$, where L follows from $V^\mu \to V'^\mu = L^\mu_{\ \nu}V^\nu$. Finally, show also that for $D_L(\Lambda) = \exp(-\eta\boldsymbol{n} \cdot \boldsymbol{\sigma}/2)$, $L[D_L(\Lambda)]$ is a boost of rapidity η (i.e. with boost parameters $\beta = \tanh\eta, \gamma = \cosh\eta$) in the direction $\boldsymbol{n}$.