Quantum Field Theory WS 2018/2019

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http://www.t75.ph.tum.de/teaching/ws18-quantum-field-theory/

Sheet 11: Lie groups, Poincaré group, spin
(16.01.2019; solution due by 23.01.2019 at 16:00, the parts required for hand-in will be announced on 22.01.2019 at 8:00am; discussed at tutorials of 23.01, 24.01 and 28.01)

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1 SU(2) and SO(3) groups [discussed at central tutorial of 23.01]

a) Show that the space of $SO(3)$ rotation matrices is isomorphic to the solid ball of radius π in three dimensions with antipodal points identified.
Hence conclude that 2π rotations around any axis $\vec{n}$ cannot be connected to the identity by a path, which can be contracted to a point. The (first) homotopy group is loosely defined as the group whose elements are the equivalence classes of paths that can be continuously deformed into each other. Explain why the homotopy group of $SO(3)$ is Z_2 . This is the reason why the rotation group allows for projective representations up to a sign. It is also the reason why there are fundamentally two types of particles, bosons and fermions.

b) Show that the universal covering group $SU(2)$ of $SO(3)$ is isomorphic to the surface of the sphere in four dimensions. What is its homotopy group?

c) Repeat the analysis of part a) for rotations in two space dimensions. What are the possible implications of your findings for spin and statistics (bosons vs. fermions) in two dimensions?

2 Lie groups [discussed at central tutorial of 23.01]

Let G be a Lie group and U some neighbourhood of e in G. We work in a chart $\eta : U \to O \subset \mathbb{R}^n$ in which the group elements have the form $g = \eta^{-1}(\varepsilon^1, \dots, \varepsilon^n)$, $e = \eta^{-1}(0, \dots, 0)$. For ε_1, $\varepsilon_2 \in O$, i.e. $\varepsilon_i = (\varepsilon_i^1, \dots, \varepsilon_i^n)$, $i = 1, 2$, we define the multiplication function

$$m(\varepsilon_1, \varepsilon_2) := \eta(\eta^{-1}(\epsilon_1)\eta^{-1}(\epsilon_2)) \tag{1}$$

which takes values in O, i.e. $m(\varepsilon_1, \varepsilon_2) = (m^1(\varepsilon_1, \varepsilon_2), \dots m^n(\varepsilon_1, \varepsilon_2))$. As shown in the lecture, the Lie algebra G' of G is spanned by the vector fields

$$X^A_{\eta^{-1}(\epsilon)}(f) = \sum_{\ell_1=1}^{n} \frac{\partial m^{\ell_1}}{\partial \varepsilon_2^A}(\epsilon, 0) \frac{\partial f \circ \eta^{-1}}{\partial \varepsilon^{\ell_1}}(\varepsilon)\,, \quad f \in C^\infty(G)\,. \tag{2}$$

Show that

$$[X^A, X^B]_e(f) = \sum_{C=1}^{n} f^{CAB} X^C_e(f) \tag{3}$$

with

$$f^{CAB} := \frac{\partial^2 m^C}{\partial \varepsilon_1^A \partial \varepsilon_2^B}(0,0) - \frac{\partial^2 m^C}{\partial \varepsilon_1^B \partial \varepsilon_2^A}(0,0). \tag{4}$$

Hints:

1) You can use that $\frac{\partial m^i}{\partial \varepsilon_1^j}(0,0) = \frac{\partial m^i}{\partial \varepsilon_2^j}(0,0) = \delta_{ij}$.

2) The notation $\frac{\partial m^i}{\partial \varepsilon_1^j}(0,0)$ etc. means you should first differentiate the function $\varepsilon_1^j \to m^i(0,\dots,\varepsilon_1^j,\dots,0)$ and only later set $\varepsilon_1^j = 0$.

3) $X^A_{\eta^{-1}(\varepsilon)}$ means the vector field X^A evaluated at $\eta^{-1}(\varepsilon) \in G$.

4) Note the notational collision $f \in C^\infty(G)$ and f^{CAB}. These are unrelated objects.

3 Poincaré group

In the Poincaré groups, spacetime translations are added to the set of (homogeneous) Lorentz transformations,

$$x^\mu \to \Lambda^\mu_{\ \nu} x^\nu + a^\mu \,, \quad a^\mu \in \mathbb{R}^4 \,, \tag{5}$$

where Λ is an element of the Lorentz group connected to the identity.

a)* [1] Prove the relations

$$U(\Lambda, a) J^{\mu\nu} U^{-1}(\Lambda, a) = \Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \sigma} \left(J^{\rho\sigma} - a^\rho P^\sigma + a^\sigma P^\rho\right) \tag{6}$$

$$U(\Lambda, a) P^\mu U^{-1}(\Lambda, a) = \Lambda^\mu_{\ \rho} P^\rho \,, \tag{7}$$

for the generators of the Poincaré algebra, $J^{\mu\nu}$ of rotations and boosts and P^μ of translations.

b)* From the previous result, derive the commutation relations of the generators $J^{\mu\nu}$, P^μ, and of $J^{\mu\nu}$ with P^μ.

c)* Prove the relations

$$U_T J^{\mu\nu} U_T^{-1} = -T^\mu_{\ \rho} T^\nu_{\ \sigma} J^{\rho\sigma} \tag{8}$$

$$U_T P^\mu U_T^{-1} = -T^\mu_{\ \rho} P^\rho \,, \tag{9}$$

for the transformation of the generators of the Poincaré algebra under time reversal. Here $T = \mathrm{diag}(-1, 1, 1, 1)$ is the time reversal transformation and U_T its anti-unitary representation on the Hilbert space.

d) Consider $P^2 = P^\mu P_\mu$ and $W^2 = W_\mu W^\mu$, where $W_\mu \equiv (1/2)\epsilon_{\mu\nu\sigma\rho} P^\nu J^{\sigma\rho}$ is the Pauli-Lubanski polarization vector. Using the commutation relations for the generators of the Poincaré group, show that P^2 and W^2 are Casimir operators for this algebra, i.e. they commute with both $J^{\mu\nu}$ and P^μ.

[1] The * means to be handed in.

e)* Since the Poincaré group has rank 2, P^2 and W^2 are its only Casimir operators. Therefore a massive state can be labelled by two numbers, its mass and spin. Show that for a massless particle, P^μ and W^μ must be proportional to each other, and use this to conclude that a massless state can be labeled by only one number (called *helicity*).

f) Show that $J^2 = J^{\mu\nu} J_{\mu\nu}$ is a Casimir of the Lorentz group, but it is not a Casimir of the Poincaré group.

4* Little group

An observer $\mathcal{O}$ sees a particle of mass m and spin $j = 1/2$ in the state $|p, s\rangle$ with momentum $p^\mu = (\sqrt{m^2 + p^2}, 0, p, 0)$ in the y-direction, where s refers to the z-component of the spin. Another observer $\mathcal{O}'$ moves relative to $\mathcal{O}$ with velocity v in the z-direction. Which state does $\mathcal{O}'$ see?

Instructions: determine the Wigner rotation $W(\Lambda, p)$ and the corresponding Wigner function $D^{\frac{1}{2}}(W)$ associated with p and the Lorentz transformation Λ which connect the frames of $\mathcal{O}$ and $\mathcal{O}'$. Use the following form of the standard boost

$$L(p)^\mu_{\ \nu} = \begin{pmatrix} \frac{p_0}{m} & \frac{p^j}{m} \\ \frac{p^i}{m} & \delta^{ij} + \left(\frac{p_0}{m} - 1\right) \frac{p^i p^j}{\vec{p}^2} \end{pmatrix}, \tag{10}$$

where p^i denotes the 3-momentum components. (Pay attention to the upper and lower index positions, which imply different signs.)

5* Poincaré group in three space-time dimensions

a) How do the commutation relations of the Poincaré algebra depend on the number of space-time dimensions? Identify the boost and rotation generators in three space-time dimensions. How many of each are there? What are the invariant (Casimir) operators in three space-time dimensions?

b) What is the little group for the massive and massless representations of the Poincaré group in three space-time dimensions?