Quantum Field Theory WS 2018/2019

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http://www.t75.ph.tum.de/teaching/ws18-quantum-field-theory/

Sheet 10: beta function, RGEs, EFT
(09.01.2019; solution due by 16.01.2019 at 16:00, the parts required for hand-in will be announced on 15.01.2019 at 8:00am; discussed at tutorials of 16.01, 17.01 and 21.01)

1*[1] Beta function

In dimensional regularization the $\overline{\text{MS}}$ coupling renormalization constant has the expansion

$$Z_\lambda = 1 + \sum_{n=1}^{\infty} \lambda^n \sum_{k=1}^{n} \frac{z_{nk}}{\epsilon^k} , \tag{1}$$

where $\lambda = \lambda(\mu)$ refers to the renormalized coupling. Prove

a) that the d-dimensional beta-function can at most be linear in ϵ,

b) that the n-loop coefficient β_{n-1} of the four-dimensional beta function,

$$\beta(\lambda) = \sum_{n=1}^{\infty} \beta_{n-1} \lambda^{n+1} , \tag{2}$$

is determined from the single pole term z_{n1}, and

c) that z_{nn} can be expressed in terms of z_{11}.

2 RGEs for two scalar fields in ϕ^4 theory

Consider a theory of two massive real scalars ϕ_1 and ϕ_2 (with standard kinetic terms) and potential

$$V(\phi_1, \phi_2) = \frac{1}{24}\lambda_1 \phi_1^4 + \frac{1}{24}\lambda_2 \phi_2^4 + \frac{1}{4}\lambda_0 \phi_1^2 \phi_2^2 , \tag{3}$$

see the discussion starting on page 168 in the lecture notes. Hence, $V(\phi_1, \phi_2)$ has three couplings λ_0, λ_1 and λ_2. (We ignore mass renormalization.)

a)* Write down all one-loop Feynman diagrams that contribute to the beta functions of λ_0, λ_1 and λ_2 in the $\overline{\text{MS}}$ scheme, and calculate these beta functions. You should find

$$\beta_{\lambda_0} = -\epsilon\lambda_0 + \frac{1}{32\pi^2}(\lambda_1 + \lambda_2)\lambda_0 + \frac{4}{32\pi^2}\lambda_0^2 , \tag{4a}$$

$$\beta_{\lambda_1} = -\epsilon\lambda_1 + \frac{3}{32\pi^2}(\lambda_1^2 + \lambda_0^2) , \tag{4b}$$

$$\beta_{\lambda_2} = -\epsilon\lambda_2 + \frac{3}{32\pi^2}(\lambda_2^2 + \lambda_0^2) . \tag{4c}$$

Hint: use the results form the lecture whenever possible (i.e. see page 188 in the lecture notes).

[1] The * means to be handed in.

b) Solve the RGEs associated to Eqs. (4) numerically.

c)* Identify the non-trivial fixed points of the RGE running and interpret them (thinking of additional symmetries).

d) Set $\lambda_2 = 0$ at the UV and see that it is generated by the RGE running from the UV to the IR.

3* EFT example

Consider the example of one light, charged scalar ϕ and a heavy real scalar S (see the lecture notes on page 180), i.e.

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi + \frac{1}{2}\partial_\mu S \partial^\mu S - \frac{1}{2} M^2 S^2 - \frac{k_3}{3!} S^3 - g S \phi^\dagger \phi \, , \tag{5}$$

with $m^2 \ll M^2$ and $k_3, g \sim O(M)$. We go to the effective field theory (EFT) of the light field ϕ at low energies (i.e. at energies $E \ll M$) by taking $p_i \cdot p_j \ll M^2$ for all four-momenta p_i in all interactions. Compute the coefficients λ_4, λ_6 and c_4 in the EFT Lagrangian of ϕ,

$$\mathcal{L}_{\text{eff}} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \frac{\lambda_4}{4} \left(\phi^\dagger \phi\right)^2 - \frac{\lambda_6}{8M^2} \left(\phi^\dagger \phi\right)^3 + \frac{c_4}{M^2} \partial_\mu (\phi^\dagger \phi) \, \partial^\mu (\phi^\dagger \phi) \, , \tag{6}$$

using

a) the tree-level diagrams of the interactions $\phi\phi \to \phi\phi$ and $\phi\phi \to \phi\phi\,\phi^\dagger\phi$ (compare also to Exercise 1 on Sheet 8), and

b) the equations of motion, i.e. use the equation of motion of the heavy field S from the Lagrangian Eq. (5) to express S in terms of the light field ϕ. Plug this expression back into the Lagrangian Eq. (5) and compare the result to the diagrammatic method.