Quantum Field Theory WS 2018/2019

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http://www.t75.ph.tum.de/teaching/ws18-quantum-field-theory/

Sheet 1: basic concepts
(24.10.2018; solution due by 31.10 at 16:00, the parts required for hand-in will be announced on 30.10 at 8:00am; discussed at tutorials of 31.10, 31.10 and 05.11)

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1 Quantization of the real (hermitian) scalar field

Consider the solution of the Klein-Gordon equation

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(e^{-ip\cdot x} a(\vec{p}) + e^{ip\cdot x} a^\dagger(\vec{p}) \right) , \tag{1}$$

where $p \cdot x = E_{\vec{p}}\, t - \vec{p} \cdot \vec{x}$ with $E_{\vec{p}} = \sqrt{m^2 + \vec{p}^{\,2}}$ and the creation $a^\dagger$ and annihilation a operators satisfy

$$[a(\vec{p}), a^\dagger(\vec{q})] = 2\, E_{\vec{p}}\, (2\pi)^3\, \delta^{(3)}(\vec{p} - \vec{q}) , \tag{2}$$

$$[a(\vec{p}), a(\vec{q})] = [a^\dagger(\vec{p}), a^\dagger(\vec{q})] = 0 . \tag{3}$$

a)*[1] Show that

$$\int dp^0 \delta(p^2 - m^2)\theta(p^0) = \frac{1}{2E_{\vec{p}}} \tag{4}$$

and therefore that

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \tag{5}$$

is a Lorentz-invariant 3-momentum integral.

b)* Derive the commutation relations for the field $\phi(x)$ and its conjugate $\pi(x) = \dot{\phi}(x)$

$$[\phi(t, \vec{x}), \pi(t, \vec{y})] = i\, \delta^{(3)}(\vec{x} - \vec{y}) , \tag{6}$$

$$[\phi(t, \vec{x}), \phi(t, \vec{y})] = [\pi(t, \vec{x}), \pi(t, \vec{y})] = 0 . \tag{7}$$

c) Express the Hamiltonian $H = P^0$ and the momentum operator P^i in terms of creation and annihilation operators.

$$P^0 = \int d^3x \left(\pi(x)\dot{\phi}(x) - \mathcal{L} \right) , \quad \mathcal{L} = \frac{1}{2}\partial_\mu\phi\, \partial^\mu\phi - \frac{m^2}{2}\phi^2 . \tag{8}$$

$$P^i = \int d^3x\, \pi(x)\partial^i\phi(x) . \tag{9}$$

d)* Show that

$$\partial^\mu\phi(x) = i[P^\mu, \phi(x)] . \tag{10}$$

[1] The * means to be handed-in.

e) Derive the following expression for the creation and annihilation operators

$$a(\vec{p}) = e^{iE_{\vec{p}}t}\left(E_{\vec{p}}\,\tilde{\phi}(t,\vec{p}) + i\,\tilde{\pi}(t,\vec{p})\right), \tag{11}$$

$$a^\dagger(\vec{p}) = e^{-iE_{\vec{p}}t}\left(E_{\vec{p}}\,\tilde{\phi}(t,-\vec{p}) - i\,\tilde{\pi}(t,-\vec{p})\right), \tag{12}$$

where $\tilde{\phi}(t,\vec{p})$ is the Fourier transform of $\phi(x)$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\,\tilde{\phi}(t,\vec{p})e^{i\vec{p}\cdot\vec{x}}, \tag{13}$$

and likewise for $\tilde{\pi}(t,\vec{p})$.

2 Noether currents of Lorentz invariance

The invariance of the relativistic distance, $ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\mu\nu}\,dx^\mu dx^\nu$, under (global) Lorentz transformations, $x^\mu \to x'^\mu = \Lambda^\mu{}_\rho x^\rho$,

$$ds^2 = \eta_{\mu\nu}\,dx^\mu dx^\nu = \eta_{\mu\nu}\,dx'^\mu dx'^\nu = \eta_{\mu\nu}\,\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\,dx^\rho dx^\sigma, \tag{14}$$

implies the condition

$$\eta_{\alpha\beta} = \eta_{\mu\nu}\,\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\,. \tag{15}$$

a) Show that infinitesimal Lorentz transformations, $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu + O(\omega^2)$, $|\omega| \ll 1$, are parametrized by an anti-symmetric matrix

$$\omega_{\mu\nu} = -\omega_{\nu\mu}\,. \tag{16}$$

How many independent generators can you identify?

b)* Show that the generators acting on a (scalar) field are given by

$$\mathcal{J}^{\alpha\beta} = i(x^\alpha\partial^\beta - x^\beta\partial^\alpha)\,. \tag{17}$$

c) Calculate the conserved currents $K^\mu_{\alpha\beta}$ associated with invariance under Lorentz transformations. Express them in terms of the energy-momentum tensor.

d)* Evaluate the currents for

$$\mathcal{L} = -\frac{1}{2}\phi\left(\Box + m^2\right)\phi\,. \tag{18}$$

Show that the rotations

$$J^i = \frac{1}{2}\varepsilon^{ijk}\int d^3x K^0_{jk} \tag{19}$$

acting on a one-particle state with zero-momentum give

$$J^i|\vec{p}=0\rangle = 0\,. \tag{20}$$

Check that the currents satisfy $\partial_\mu K^\mu_{\ \alpha\beta} = 0$ on the equations of motion.

e)* What is the physical interpretation of the conserved quantities Q_i associated with boosts?

$$Q_i = \int d^3x K^0_{\ i0} \, . \tag{21}$$

f) Show that $\frac{dQ_i}{dt} = 0$ can still be consistent with $i\frac{\partial Q_i}{\partial t} = [Q_i, H]$. Thus, although these charges are conserved, they do not provide invariants for the equations of motion. This is one way to understand why particles have spin, corresponding to representations of the rotation group, and not additional quantum numbers associated with boosts.