Quantum Field Theory WS 2018/2019

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Sheet 0: introduction
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1 Functional derivative

For a functional $F(\phi)$ acting on a function $\phi(x)$, the functional derivative $\delta F[\phi]/\delta\phi(x)$ is defined via

$$F[\phi+\eta] = F[\phi] + \int dx' \frac{\delta F[\phi]}{\delta\phi(x')}\eta(x') + \ldots$$

where $\eta(x)$ is an infinitesimal function, and the dots stand for terms of higher order in η. In particular, for the functional $F[\phi] = \phi(x)$,

$$\frac{\delta\phi(x)}{\delta\phi(x')} = \delta(x-x').$$

a) Derive from the definition above the product rule for the functional derivative,

$$\frac{\delta(F[\phi]G[\phi])}{\delta\phi(x)} = \frac{\delta F[\phi]}{\delta\phi(x)}G[\phi] + F[\phi]\frac{\delta G[\phi]}{\delta\phi(x)}\,. \tag{1}$$

b) Prove also the chain rule

$$\frac{\delta F[G[\phi]]}{\delta\phi(x)} = \int dy \frac{\delta F[G]}{\delta G(y)}\frac{\delta G[\phi]}{\delta\phi(x)}\,, \tag{2}$$

where $G : \phi(x) \to G(y)$ associates a function $G(y)$ to a function $\phi(x)$: therefore, for fixed y, G is a functional $G[\phi]$, and for fixed ϕ, G is a function $G(y)$, for example

$$G(y) = \int dx K(x-y)\phi(x)\,, \qquad F[G] = \int dy (G(y))^2\,. \tag{3}$$

c) Show that the Euler-Lagrange equations are equivalent to the functional equations for the action

$$\frac{\delta S[\phi_n]}{\delta\phi_n(x)} = 0 \qquad \text{with} \quad \phi_n = \phi, \partial_\mu\phi, \ldots. \tag{4}$$

d) Use functional derivatives to obtain the equation of motion for the field ϕ from

$$S = \int d^4x \big[\dot\phi^2(x) - a\phi^2(x) - b\phi^4(x) - c^2(\nabla\phi(x))^2\big], \tag{5}$$

where a, b, c are constants.

e) Repeat the previous point after replacing the quantity in square parentheses with

$$\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \lambda\,\phi^2(\partial_\mu\phi)^4 \tag{6}$$

where λ is a constant. What is its mass dimension?

2 Fock space

An N-particle momentum eigenstate is defined as

$$|p_1 \dots p_N\rangle = \left(\prod_i \frac{1}{\sqrt{n_i(p)!}} \right) a^\dagger(p_1) \dots a^\dagger(p_N)|\Omega\rangle \tag{7}$$

where $n_i(p)$ denotes the number of occurrences of p in the set $\{p_1, \dots p_N\}$ and the product runs over all distinct values of p in this set. The union of all momentum eigenstates for any N defines a basis of the Fock space, on which a real scalar quantum field acts.

a) Prove that

$$a^\dagger(p)|p_1 \dots p_N\rangle = \sqrt{n(p)+1}\,|pp_1 \dots p_N\rangle \tag{8}$$

$$a(p)|p_1 \dots p_N\rangle = \frac{1}{\sqrt{n(p)}} \sum_{n=1}^{N} (2\pi)^3 2E_p \delta^{(3)}(\vec{p} - \vec{p_n})|p_1 \dots p_{n-1}p_{n+1} \dots p_N\rangle \tag{9}$$

where the last expression should be interpreted as zero, if p does not coincide with any of the p_n .

b) Show that the state has total momentum $p_1 + \cdots + p_N$.

c) Show that the state is totally symmetric under the interchange of any two momentum labels.

d) Construct explicitly the single-particle momentum eigenstate wave function $\psi(t, \vec{x}) = \langle\Omega|\phi|p\rangle$. Verify that $|\vec{x}, t\rangle \equiv \phi(t, \vec{x})|\Omega\rangle$ can be interpreted as an eigenstate of the position operator for a quantum-mechanical system of a single particle in the Schrödinger picture.
Compute the wave-function $\psi(t, \vec{x}, \vec{y}) = \langle\vec{x}\vec{y}; t|p_1p_2\rangle$ of a two-particle momentum eigenstate.

e) Prove the identity

$$\langle p'_1 \dots p'_N|p_1 \dots p_N\rangle = \left(\prod_i \frac{1}{\sqrt{n_i(p)!}} \right) \left[\prod_{n=1}^{N} (2\pi)^3 2E_{p_n} \delta^{(3)}(\vec{p'_n} - \vec{p_n}) + \text{permutations} \right]. \tag{10}$$