# Advanced Quantum Field Theory SS 2019

Prof. Andreas Weiler (TUM); Dr. Ennio Salvioni, Dr. Javi Serra
https://www.t75.ph.tum.de/teaching/ss19-quantum-field-theory-ii/

**Bonus sheet: amplitudes, QCD chiral Lagrangian** (26.07.2019)

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## 1 Amplitudes in gauge theory and gravity

**a)** Consider the process $e_L^-(p_1)e_R^+(p_2) \to \gamma(p_3)\gamma(p_4)$, with amplitude

$$i\mathcal{M} = (-ie)^2 \langle 2|\, \not{\epsilon}_4 \frac{i(\not{p}_2 + \not{p}_4)}{s_{24}} \not{\epsilon}_3 |1] + (3 \leftrightarrow 4)\ , \tag{1}$$

and show that $\mathcal{M}(e_L^- e_R^+ \gamma_L \gamma_L) = 0$ without making any special choices of the reference spinors $r_3$ and $r_4$.

**b)** For the same process, show that the amplitude $\mathcal{M}(e_L^- e_R^+ \gamma_R \gamma_L)$ is independent of $r_3$ and $r_4$ by deriving it without making a special choice for the reference spinors.

**c)** Consider the 4-particle (massless) amplitude (all particles outgoing)

$$\mathcal{M}_4 = g \frac{\langle 12\rangle^7 [12]}{\langle 13\rangle \langle 14\rangle \langle 23\rangle \langle 24\rangle \langle 34\rangle^2}\ , \tag{2}$$

and figure out the helicity of the particles, the dimension of the coupling $a$, and the theory that could produce such an amplitude.

**d)** Construct all the possible 3-particle amplitudes associated with massless (spin-2) gravitons, argue which ones are unphysical, and identify the ones associated with the Einstein-Hilbert action $\frac{M_{Pl}^2}{2}\int d^4x\sqrt{-g}R$.

**e)** Consider the dimension-5 Higgs-gluon fusion operator $h\mathrm{Tr}[G_{\mu\nu}G^{\mu\nu}]$. Use little group scaling to determine the 3-particle amplitudes of this operator in the limit of $m_h = 0$.

**f)** Calculate the 4-graviton amplitude $\mathcal{M}_4(1_L 2_L 3_R 4_R)$, recalling first the 3-particle amplitudes associated to the Einstein-Hilbert action and then using the $[1,2\rangle$-shift BCFW recursion relation (you can find examples of this method in 1308.1697, section 3.2):

$$|1\rangle \to |1\rangle, \quad |1] \to |1] + z|2], \quad |2\rangle \to |2\rangle - z|1\rangle, \quad |2] \to |2]. \tag{3}$$

Check the little group scaling and Bose-symmetry of your answer and show that $\mathcal{M}_4(1_L 2_L 3_R 4_R)$ obeys the 4-point "KLT" relation[1]

$$\mathcal{M}_4(1_L 2_L 3_R 4_R) = s_{12} A_4[1_L 2_L 3_R 4_R] A_4[1_L 2_L 3_R 4_R]\ , \tag{4}$$

where $A_4[1_L 2_L 3_R 4_R]$ is the MHV 4-gluon amplitude (known as Parke-Taylor),

$$A_4[1_L 2_L 3_R 4_R] = \frac{\langle 12\rangle^4}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle}\ . \tag{5}$$

which we have seen in the lecture.[2]

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[1] The Kawai-Lewellen-Tye (KLT) relations derived from string theory tell us that perturbative gravity amplitudes are the "square" of the corresponding amplitudes in gauge theory. H. Kawai, D. C. Lewellen and S. H. H. Tye, "A Relation Between Tree Amplitudes of Closed and Open Strings," Nucl. Phys. B **269** (1986)

[2] Compare to how difficult the same calculation is with Feynman diagrams: S. Sannan, "Gravity as the Limit of the Type II Superstring Theory," Phys. Rev. D **34** (1986) 1749.

## 2 The QCD chiral Lagrangian

Consider the low-energy description of 3-flavor QCD, where the chiral $SU(3)_L \times SU(3)_R$ symmetry is spontaneously broken to the diagonal subgroup. We parametrize the 8 associated Goldstone bosons with the field $\Sigma = \exp(2i\Pi/f)$, where $\Pi \equiv \pi^a \lambda^a$ with $\lambda^a$ the Gell-Mann matrices, that transforms as $\Sigma \to L\Sigma R^\dagger$. Explicitly, we write the light fields as

$$\frac{\Pi}{\sqrt{2}} = \begin{pmatrix} \pi_3/\sqrt{2} + \pi_8/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\pi_3/\sqrt{2} + \pi_8/\sqrt{6} & K^0 \\ K^- & \overline{K}^0 & -\sqrt{2/3}\,\pi_8 \end{pmatrix} \tag{6}$$

where $\overline{K}^0 = K^{0\dagger}$. The unique operator at the leading-order in derivatives is $\mathcal{L}_2 = (f^2/16)\mathrm{Tr}[(\partial_\mu \Sigma)^\dagger \partial^\mu \Sigma]$.

**a)** The non-vanishing masses for $u, d, s$ break the chiral symmetry explicitly. To account for this fact in the chiral Lagrangian we introduce a spurion $M$ that formally transforms as $M \to LMR^\dagger$, in such a way that the quark mass term $\bar{q}_L M q_R + \text{h.c.}$ is formally invariant. Argue that, at the linear order in $M$ and without derivatives, there is only one invariant operator that we can add, namely $\delta\mathcal{L}_M = (Bf^2/8)\mathrm{Tr}(M^\dagger \Sigma + \Sigma^\dagger M)$ where $B$ (sometimes also called $m_0$) is a constant with mass dimension 1. This method ensures that chiral symmetry is broken in the effective theory in exactly the same way as in QCD. Set the spurion to its physical value $M = \mathrm{diag}\,(m_u, m_d, m_s)$ and calculate the Goldstone masses that arise from $\delta\mathcal{L}_M$. Observe that $\pi_3$ and $\pi_8$ mix, but it is not necessary to calculate their eigenvectors $\pi^0$ and $\eta$ and the corresponding eigenvalues, to arrive at the relation

$$m_\eta^2 + m_{\pi^0}^2 = \frac{2}{3}(m_{K^0}^2 + m_{K^+}^2 + m_{\pi^+}^2). \tag{7}$$

**b)** Derive the ratios of quark masses

$$\frac{m_u}{m_d} = \frac{m_{K^+}^2 - m_{K^0}^2 + m_{\pi^+}^2}{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2} \approx 0.67\,, \qquad \frac{m_s}{m_d} = \frac{m_{K^0}^2 + m_{K^+}^2 - m_{\pi^+}^2}{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2} \approx 20\,. \tag{8}$$

Armed with these results, go back to the $\pi_3$-$\pi_8$ mixing, diagonalize it and evaluate $m_{\pi^+} - m_{\pi^0}$. Compare the result to the experimental value $m_{\pi^+} - m_{\pi^0} = (139.6 - 135.0)$ MeV $= 4.6$ MeV.

**c)** The correction $\propto (m_u - m_d)$ discussed in the previous point is far too small to explain the observed splitting among the pions. This splitting is in fact dominantly caused by electromagnetism. To include its effects, we proceed again with spurions: write the coupling of the photon to the quarks as $\overline{q}_L Q_L \not{A} q_L + \overline{q}_R Q_R \not{A} q_R$, with formal transformation properties $Q_L \to L Q_L L^\dagger$ and $Q_R \to R Q_R R^\dagger$. Argue that at the quadratic order in $Q_{L,R}$ and zero derivatives, there is only one operator we can consider, $\delta\mathcal{L}_Q = C\mathrm{Tr}(\Sigma^\dagger Q_L \Sigma Q_R)$ where $C$ is a constant of mass dimension 4. Then, set the spurions to the physical values $Q_L = Q_R = e\,\mathrm{diag}\,(2/3, -1/3, -1/3)$ and show that this operator gives (neglecting a constant),

$$\delta\mathcal{L}_Q = -\frac{8e^2 C}{f^2}(K^+K^- + \pi^+\pi^-) + O(1/f^4), \tag{9}$$

that is, it contributes equally to the masses of the charged pion and kaon. Rederive the first of Eqs. (8) including the electromagnetic correction just computed (and neglecting the small $\pi_3$-$\pi_8$ mixing), and show that it now reads

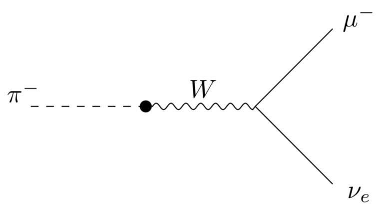
$$\frac{m_u}{m_d} = \frac{m_{K^+}^2 - m_{K^0}^2 + 2m_{\pi^0}^2 - m_{\pi^+}^2}{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2} \approx 0.56\,. \tag{10}$$

**d)** For simplicity, in the rest of the exercise we restrict our attention to the 2-flavor chiral Lagrangian, where $\Sigma = \exp(2i\pi^a\sigma^a/f_\pi)$ with $\sigma^a$ the Pauli matrices. We aim to include the electroweak interactions. We do so by promoting the $SU(2)_L \times U(1)_Y$ subgroup of $SU(2)_L \times SU(2)_R$ to a gauge symmetry: the two-derivative Lagrangian becomes

$$\mathcal{L}_2 = \frac{f_\pi^2}{16}\mathrm{Tr}[(D_\mu\Sigma)^\dagger D^\mu\Sigma], \qquad D_\mu\Sigma = \partial_\mu\Sigma - ig\frac{\sigma^a}{2}W_\mu^a\Sigma + ig'\Sigma\frac{\sigma^3}{2}B_\mu\,. \tag{11}$$

Set $\Sigma$ to its expectation value $\langle\Sigma\rangle = \mathbf{1}_2$ and calculate the spectrum of gauge boson masses that arises from Eq. (11). Knowing that experimentally $g \approx 2/3$ and $f_\pi \approx 186$ MeV, evaluate the $W$ mass originating from chiral symmetry breaking in QCD. Show that it is a factor $\sim 3000$ smaller than the observed value $\approx 80$ GeV. Calculate also the quantity $\rho = m_W^2/(m_Z^2\cos^2\theta_w)$.

**e)** Finally, we calculate the lifetime for the decay of the charged pion $\pi^+$ to $\mu^+\nu_\mu$. From Eq. (11) extract the piece that mixes $\pi^+$ and $W$, of the form $\sim \partial_\mu\pi^+W^{-\mu} + \text{h.c.}$. Using also the expression of the interaction between the $W$ and the leptons, $(g/\sqrt{2})\,\overline{\mu}\,\not{W}^-P_L\nu_\mu + \text{h.c.}$, calculate the amplitude for the following diagram




and square it, obtaining $\sum_{\text{spins}}|\mathcal{M}|^2 = f_\pi^2G_F^2m_\mu^2m_\pi^2(1 - m_\mu^2/m_\pi^2)$. Explain the physical reason why it vanishes for $m_\mu \to 0$. Then calculate the total width for the decay, for which you should find

$$\Gamma(\pi^+ \to \mu^+\nu_\mu) = \frac{m_\pi}{16\pi}f_\pi^2G_F^2m_\mu^2(1 - m_\mu^2/m_\pi^2)^2. \tag{12}$$

Here $G_F$ is the Fermi constant; note that conventions for $f_\pi$ differ in the literature, for example $f_\pi(\text{us}) = 2f_\pi(\text{Peskin})$.