Advanced Quantum Field Theory SS 2019

Prof. Andreas Weiler (TUM); Dr. Ennio Salvioni, Dr. Javi Serra

https://www.t75.ph.tum.de/teaching/ss19-quantum-field-theory-ii/

Sheet 8: SM and trace anomalies, sigma model (12.07.2019)

---

1 Anomalies in the Standard Model

a) Assigning arbitrary hypercharges a, b, c, d and e to the Standard Model (SM) multiplets (u_L, d_L), u_R, d_R, (ν_L, e_L) and e_R, respectively, compute the anomaly coefficients for the SM currents associated with the $SU(3)_{\text{QCD}} \times SU(2)_{\text{weak}} \times U(1)_Y$ gauge symmetries, including gravitational anomalies. Check that the requirement of vanishing anomalies gives rise to four relevant equations, which have only two solutions, one of them corresponding to the hypercharge assignments in the SM. Is it possible then to couple a gauge boson to any other $U(1)$ quantum number in addition to hypercharge?

Hint: Recall that the anomaly coefficients are given by

$$\mathcal{A}^{abc} = \text{Tr}[t^a\{t^b, t^c\}] \,, \tag{1}$$

where t^a are the gauge symmetry generators. Besides, the only relevant gravitational anomaly is that associated with hypercharge.

b) Are the baryon number (B) and lepton number (L) currents anomaly free? What about $B + L$, $B - 3L_a \, (a = e, \mu, \tau)$, $L_a - L_b$ or $B - L$?

c) Think of the minimal extension of the SM that makes $B - L$ anomaly-free, even considering gravitational anomalies.

2 Trace anomaly

Consider a Yang-Mills theory with a massive scalar in the fundamental representation,

$$S = \int d^4x \sqrt{g} \left[-\frac{1}{4} g^{\mu\lambda} g^{\nu\sigma} F^a_{\mu\nu} F^a_{\lambda\sigma} + g^{\mu\nu} D_\mu \phi^\dagger D_\nu \phi - m^2 \phi^\dagger \phi \right] \,. \tag{2}$$

a) Explicitly derive the energy-momentum tensor of the (classical) theory, defined by considering the infinitesimal variation of the metric

$$S(g^{\mu\nu}) \to S(g^{\mu\nu} + \delta g^{\mu\nu}) = S(g^{\mu\nu}) + \delta S \tag{3}$$

under which

$$\delta S = \frac{1}{2} \int d^4x \sqrt{g}\, \Theta_{\mu\nu} \delta g^{\mu\nu} \,. \tag{4}$$

Check that its trace, Θ^μ_μ, vanishes in the limit $m \to 0$.

b) Once quantum corrections are included, does Θ^μ_μ still vanish when $m \to 0$?

c) Scale transformations (or dilatations) are given by

$$x \to e^\lambda x \,, \tag{5}$$

while fields also transform according to their scaling dimension, e.g. for a scalar field,

$$\phi(x) \to e^{\lambda}\phi(e^{\lambda}x). \tag{6}$$

In analogy with the spontaneous breaking of global symmetries, if scaling symmetry is spontaneously broken one should expect a Goldstone boson, σ (known as a dilaton), which transforms under dilatations

$$\sigma(x) \to \sigma(e^{\lambda}x) + \lambda\ . \tag{7}$$

Derive the interactions of the dilaton to the gauge fields and scalar in (2) by the requirement that dilatations are in fact a good (but "hidden") symmetry of the system. In other words, amend the lagrangian (2) by adding terms $\Delta\mathcal{L}_{\rm int}(\sigma)$ such that the action is scale invariant.

Hint: Include the effect of the β-function. You can check that your result agrees with the expectation for Goldstone bosons

$$\Delta\mathcal{L}_{\rm int}(\sigma) = -\sigma\partial_\mu D^\mu \tag{8}$$

where D^μ is the dilatation current.

3 Representation independence of sigma model

Consider the $SU(2)_L \times SU(2)_R$ invariant sigma model, whose Lagrangian in the *linear representation* is

$$\mathcal{L}_1 = \frac{1}{4}{\rm Tr}\,[(\partial_\mu\Sigma)^\dagger\partial^\mu\Sigma] + \frac{\mu^2}{4}{\rm Tr}\,[\Sigma^\dagger\Sigma] - \frac{\lambda}{16}\big[{\rm Tr}\,\Sigma^\dagger\Sigma\big]^2, \qquad \Sigma = \varphi\,\mathbf{1}_2 + i\,\vec{\pi}\cdot\vec{\sigma} \tag{9}$$

with σ^a $(a = 1,2,3)$ the Pauli matrices. Under $SU(2)_L \times SU(2)_R$, the sigma field transforms as $\Sigma \to L\Sigma R^\dagger$. Assume $\mu^2 > 0$ and $\lambda > 0$, so that the symmetry is spontaneously broken to the diagonal $SU(2)_V$ by the vacuum expectation value $\langle\varphi\rangle = \sqrt{\mu^2/\lambda} \equiv v$. Define the physical excitation $\tilde{\varphi} = \varphi - v$.

a) By defining the variables

$$S \equiv \sqrt{(v+\tilde{\varphi})^2 + \vec{\pi}^{\,2}} - v = \tilde{\varphi} + \ldots\ , \qquad \vec{\Phi} \equiv \frac{v}{\sqrt{(v+\tilde{\varphi})^2 + \vec{\pi}^{\,2}}}\,\vec{\pi} = \vec{\pi} + \ldots \tag{10}$$

where we show the leading order in the $1/v$ expansion, prove that the Lagrangian can be written in the form

$$\mathcal{L}_2 = \frac{1}{2}\left[(\partial_\mu S)^2 - 2\mu^2 S^2\right] - \lambda v S^3 - \frac{\lambda}{4}S^4 + \frac{1}{2}\left(1+\frac{S}{v}\right)^2\left[(\partial_\mu\vec{\Phi})^2 + \frac{(\vec{\Phi}\cdot\partial_\mu\vec{\Phi})^2}{v^2 - \vec{\Phi}^2}\right] \tag{11}$$

which we call the *square root representation.*

b) A third form $\mathcal{L}_3$ of the Lagrangian, called *exponential representation*, is obtained by substituting

$$\Sigma = (v+\rho)U, \qquad U = \exp(i\,\vec{\pi}'\cdot\vec{\sigma}/v) \tag{12}$$

in the linear Lagrangian $\mathcal{L}_1$ defined in Eq. (9). Calculate the explicit expression of $\mathcal{L}_3$.

c) Verify, by looking at terms bilinear in the fields, that the three representations $\mathcal{L}_i$, $i = 1, 2, 3$, all have the same spectrum of free particles.

d) Calculate in all three parameterizations, at tree level and up to $O(p^2)$, the amplitude for the elastic scattering of one neutral and one charged Goldstone boson, $\pi^+\pi^0 \to \pi^+\pi^0$. For $\mathcal{L}_1$ the neutral GB is $\pi^0 = \pi^3$ and the charged GB is $\pi^\pm = (\pi^1 \mp i\pi^2)/\sqrt{2}$, and similarly for $\mathcal{L}_{2,3}$. Observe that you find the same answer in all cases.